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From G-parking functions to B-parking functions*

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Abstract

A matching M in a multigraph $G = (V, E)$ is said to be uniquely restricted if M is the only perfect matching in the subgraph of G induced by $V(M)$ (i.e., the set of vertices saturated by M). For any fixed vertex x_0 in G, there is a bijection from the set of spanning trees of G to the set of uniquely restricted matchings of size $|V| - 1$ in $S(G) - x_0$, where $S(G)$ is the bipartite graph obtained from G by subdividing each edge in G. Thus the notion "uniquely restricted matchings of a bipartite graph H saturating all vertices in a partite set X" can be viewed as an extension of "spanning trees in a connected graph". Motivated by this observation, we extend the notion "G-parking functions" of a connected multigraph to "B-parking functions" $f : X \to \{-1, 0, 1, 2, \cdots\}$ of a bipartite graph H with a bipartition (X, Y) and find a bijection ψ from the set of uniquely restricted matchings of H to the set of B-parking functions of H. We also show that for any uniquely restricted matching M in H with $|M| = |X|$, if $f = \psi(M)$, then $\sum_{x\in X} f(x)$ is exactly the number of elements $y \in Y - V(M)$ which are not externally B-active with respect to M in H, where the new notion "externally B-active members with respect to M in H" is an extension of "externally active edges with respect to a spanning tree in a connected multigraph".

MSC: 05A19, 05B35 and 05C85

Keywords: graph, spanning tree, parking function, bijection

1 Introduction

The notion of a parking function was introduced by Konheim and Weiss [11] in 1966. Suppose that there are n drivers labeled $1, 2, \cdots, n$ and n parking spaces arranged in a line numbered $1, 2, \cdots, n$. Assume that these n drivers enter the parking area in the order $1, 2, \cdots, n$ and driver i parks at space j, where j is the minimum number with $f(i) \le j \le n$ such that space j is unoccupied by the previous drivers and $f(i)$ is the

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initial parking preference of driver i. If all drivers can park successfully by this rule, then $(f(1), f(2), \cdots, f(n))$ is called a *parking function* of length n. Mathematically, a function $f : N_n \to N_n$, where $N_n = \{1, 2, \cdots, n\}$, is called *a parking function* if the inequality $|\{1 \le i \le n : f(i) \le k\}| \ge k$ holds for each integer $k : 1 \le k \le n$. For example, for $n = 2$, $(f(1), f(2)) = (1, 1)$, $(f(1), f(2)) = (1, 2)$ and $(f(1), f(2)) = (2, 1)$ are parking functions, but $(f(1), f(2)) = (2, 2)$ is not. It can be shown easily that $f : N_n \to N_n$ is a parking function if and only if there is a permutation $\pi_1, \pi_2, \cdots, \pi_n$ of N_n such that $f(\pi_j) \le j$ holds for all $j = 1, 2, \cdots, n$. Konheim and Weiss [11] proved that that the number of parking functions of length n is equal to $(n+1)^{n-1}$, which is equal to the number of spanning trees of the complete graph K_{n+1} ([1, 3]).

The parking function and its various extensions have been studied by many researchers [4, 5, 7, 9, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 30]. One of the extensions, due to Postnikov and Shapiro [18], was from parking functions to G-parking functions for connected multigraphs without loops.

Before talking about G-parking functions, let's first introduce the notations of graphs used in this article. Unless stated otherwise, we always assume that

(i) $G = (V, E)$ is a connected multigraph without loops, where $V = \{x_0, x_1, \cdots, x_n\}$ and $E = \{y_1, y_2, \cdots, y_m\}$. For any non-empty subsets V' of V and E' of E, let $G[V']$ and $G[E']$ be the subgraphs of G induced by V' and E' respectively;

(ii) H is a simple and bipartite graph with a bipartition (X, Y), where $X = \{x_1, x_2, \cdots, x_n\}$ and $Y = \{y_1, y_2, \cdots, y_m\}$; and

(iii) H_{G,x_0} is the special bipartite graph $S(G) - x_0$ with a bipartition (X, Y), where x_0 is a fixed vertex in G, $S(G)$ is obtained from G by subdividing each edge in G, $X = V - \{x_0\}$ and $Y = E$. An example of H_{G,x_0} is shown in Figure 1.

Both graphs G and H have fixed weight functions which are used for comparing edges in G or elements of Y in H. The weight function for G is an injective mapping $w : E \to \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of non-negative integers, while the weight function for H is an injective mapping $w : Y \to \mathbb{N}_0$. Thus the weight function w of G is also the weight function of H_{G,x_0}. The mapping w is injective in order to distinguish $w(y_1)$ and $w(y_2)$ for any distinct elements y_1 and y_2.

For any subsets V_1 and V_2 of V, let $E_G(V_1, V_2)$ denote the set of those edges in G joining a vertex in V_1 and a vertex in V_2. In particular, let $E_G(u, V_2) = E_G(\{u\}, V_2)$ for any $u \in V$. So $d_G(u) = |E_G(u, V)|$ is the degree of vertex u in G. A function $f : V - \{x_0\} \to \mathbb{N}_0$ is called a *G-parking function* with respect to x_0 if for any non-empty subset $V' \subseteq V - \{x_0\}$, there exists $u \in V'$ with $|E_G(u, V - V')| > f(u)$. Let $\mathcal{GP}(G, x_0)$ denote the set of G-parking functions of G with respect to x_0.

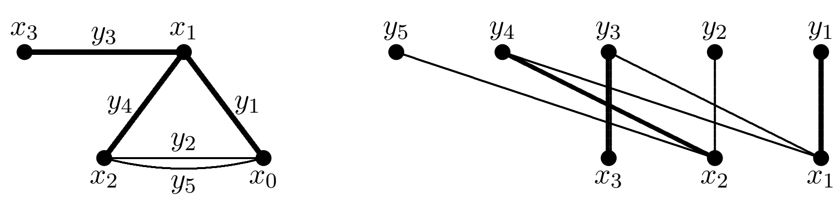
By Corollary 2.6, which was due to Dhar [6], a function $f : V - \{x_0\} \to \mathbb{N}_0$ belongs to $\mathcal{GP}(G, x_0)$ if and only if there is an ordering $x_{\pi_1}, x_{\pi_2}, \cdots, x_{\pi_n}$ of vertices in $V - \{x_0\}$ such that $|E_G(x_{\pi_i}, V - V_i)| > f(x_{\pi_i})$ holds for all $i = 1, 2, \cdots, n$, where $V_i = \{x_{\pi_j} : i \le j \le n\}$. Hence a function $f : N_n \to N_n$ is a parking function of length n if and only if $f - 1 \in \mathcal{GP}(K_{n+1}, 0)$, where $V(K_{n+1}) = \{0, 1, 2, \cdots, n\}$.

The most interesting property on G-parking functions is the existence of bijections from the set of spanning trees of G, denoted by $\mathcal{T}(G)$, to $\mathcal{GP}(G, x_0)$. Several such bijections have been obtained (see [5] for example).

In this paper, we focus on presenting a new extension of G-parking functions.

A matching M of a graph G is said to be *uniquely restricted* (UR) if M is the only perfect matching in $G[V(M)]$, where $V(M)$ is the set of vertices saturated by edges in M. Clearly, a matching M of G is a UR-matching if and only if $|E(C)| > 2|E(C) \cap M|$ holds for every cycle C in G, where $E(C)$ is the set of edges on C. The notion of UR-matchings was first introduced by Golumbic, Hirst, and Lewenstein [8], originally motivated by the problem of determining a lower bound on the rank of a matrix having a specified zero/non-zero pattern. They [8] showed that the problem of finding a UR-matching with the maximum cardinality in an input graph is known to be NP-complete even for the special cases of split graphs and bipartite graphs.

For any $T \in \mathcal{T}(G)$ with $E(T) = \{y_{\tau_i} : i = 1, 2, \cdots, n\}$, let M_T denote the matching $\{x_{\pi_i} y_{\tau_i} : i = 1, 2, \cdots, n\}$ of H_{G,x_0}, where x_{π_i} is the end of edge y_{τ_i} in G such that y_{τ_i} is contained in the unique path of T connecting x_0 and x_{π_i}. An example of T and M_T is shown in Figure 1. Proposition 2.4 shows that the mapping λ defined by $\lambda(T) = M_T$ is a bijection from $\mathcal{T}(G)$ to the set of UR-matchings of size n $(= |V| - 1)$ in H_{G,x_0}.



(a) G (b) Graph H_{G,x_0}

Figure 1: $E(T) = \{y_1, y_3, y_4\}$ and $M_T = \{x_1 y_1, x_3 y_3, x_2 y_4\}$

The above observation shows that the notion "a spanning tree of a connected multigraph" can be viewed as a special case of the notion "a UR-matching of size $|X|$ in a bipartite graph H". Motivated by this relation, we extend the notion of G-parking functions of connected multigraphs to that of B-parking functions of bipartite graphs.

Let $\mathcal{UM}(H)$ be the set of UR-matchings of H. For any $S \subseteq X$, let $\mathcal{UM}_S(H)$ be the set of those members M of $\mathcal{UM}(H)$ with $V(M) \cap X = S$. In particular, $\mathcal{UM}_X(H)$ is the set of those members M of $\mathcal{UM}(H)$ with $X \subseteq V(M)$. Thus $\mathcal{UM}(H)$ can be partitioned into subsets $\mathcal{UM}_S(H)$ for all subsets S of X.

A mapping $f : X \to \{-1\} \cup \mathbb{N}_0$ is called a *B-parking function* of H at X if for any non-empty subset S of $X_{(f\geq 0)}$, where $X_{(f\geq 0)} = \{x \in X : f(x) \geq 0\}$, there exists $x' \in S$ such that x' has at least $f(x')+1$ neighbors of degree 1 (i.e., leaves) in the subgraph of H induced by $\bigcup_{x\in S} N_H[x]$, where $N_H(x)$ is the set of neighbors of x in H and $N_H[x] = \{x\} \cup N_H(x)$. Let $\mathcal{BP}(H)$ be the family of B-parking functions of H at X. For any $S \subseteq X$, let $\mathcal{BP}_S(H)$ be the set of those members $f \in \mathcal{BP}(H)$ with $X_{(f\geq 0)} = S$. In particular, $\mathcal{BP}_X(H)$ is the set of those members $f \in \mathcal{BP}(H)$ with $f(x) \geq 0$ for all $x \in X$. Thus $\mathcal{BP}(H)$ is also partitioned into subsets $\mathcal{BP}_S(H)$ for all subsets $S \subseteq X$.

In Section 2, we give some basic properties on members in $\mathcal{UM}_X(H)$ and members in $\mathcal{BP}_X(H)$. Proposition 2.3 shows that $\mathcal{UM}_X(H) = \emptyset$ if and only if $\mathcal{BP}_X(H) = \emptyset$. Proposition 2.4 shows that the members in $\mathcal{T}(G)$ correspond to members in $\mathcal{UM}_X(H_{G,x_0})$ and Proposition 2.5 shows that $\mathcal{GP}(G, x_0) = \mathcal{BP}_X(H_{G,x_0})$.

In Section 3, we design an algorithm, called Algorithm A, for any input (H, Y'), where $Y' \subseteq Y$. Whenever $\mathcal{UM}_X(H[X \cup Y']) \neq \emptyset$, running this algorithm outputs a permutation $\pi_1, \pi_2, \cdots, \pi_n$ of $1, 2, \cdots, n$, an n-permutation $\tau_1, \cdots, \tau_n$ of $1, 2, \cdots, m$ and subsets $D(x_{\pi_i})$ of $Y - Y'$ for $i = 1, 2, \cdots, n$. In this case, the mapping $f : X \to \mathbb{N}_0$ defined by $f(x_{\pi_i}) = |D(x_{\pi_i})|$ for $i = 1, 2, \cdots, n$ is a member in $\mathcal{BP}_X(H)$. This result yields a mapping ψ_H from $\mathcal{UM}_X(H)$ to $\mathcal{BP}_X(H)$. The outputs π_i, τ_i and $D(x_{\pi_i})$ for $i = 1, 2, \cdots, n$ of running Algorithm A provide information for interpreting members in $\mathcal{BP}_X(H)$.

In Section 4, we show that the mapping ψ_H from $\mathcal{UM}_X(H)$ to $\mathcal{BP}_X(H)$, defined by $\psi_H(M) = f$, is a bijection, where f is the mapping from X to $\mathbb{N}_0$ defined by $f(x_{\pi_i}) = |D(x_{\pi_i})|$ for all $i = 1, 2, \cdots, n$, and π_i and $D(x_{\pi_i})$ are outputs of running Algorithm A with input $(H, V(M) \cap Y)$. Clearly, $\psi_{H[N[S]]}$ provides a bijection from $\mathcal{UM}_S(H)$ to $\mathcal{BP}_S(H)$ for every $S \subseteq X$, where $N[S] = \bigcup_{x\in S} N_H[x]$. Thus, there is a bijection from $\mathcal{UM}(H)$ to $\mathcal{BP}(H)$. When H is the graph H_{G,x_0}, ψ_H is a bijection ϕ_G from $\mathcal{T}(G)$ to $\mathcal{GP}(G, x_0)$ for any connected multigraph G, where $x_0 \in V(G)$.

In Section 5, we introduce the new notion "externally B-active members with respect to M in H", where $M \in \mathcal{UM}_X(H)$, defined in Page 23, which is an extension of "externally active edges with respect to a spanning tree T in a connected multigraph" defined by Tutte [28]. For any $M \in \mathcal{UM}_X(H)$, if $f = \psi_H(M)$, then $f(x_{\pi_i})$ is interpreted as the number of those $y \in N_H(x_{\pi_i}) - \big(V(M) \cup \bigcup_{s>i} N_H(x_{\pi_s})\big)$ which are not externally B-active with respect to M in H, implying that $\sum_{x_i \in X} f(x_i)$ is exactly the number of those vertices $y \in Y - V(M)$ which are not externally B-active with respect to M in H. This result implies that there exists a bijection ϕ_G from $\mathcal{T}(G)$ to $\mathcal{GP}(G, x_0)$ such that for any $T \in \mathcal{T}(G)$, if $f = \phi_G(T)$, then $\sum_{x \in V(G)-\{x_0\}} f(x)$ is exactly the number of those edges in $E(G) - E(T)$ which are not externally active with respect to T.

2 UR-matchings and B-parking functions

In this section, we characterize UR-matchings and B-parking functions of a bipartite graph H. It is proved in Proposition 2.3 that $\mathcal{UM}_X(H) = \emptyset$ if and only if $\mathcal{BP}_X(H) = \emptyset$. For the special bipartite graph H_{G,x_0}, Propositions 2.4 and 2.5 show that $\mathcal{T}(G)$ and $\mathcal{GP}(G, x_0)$ correspond to $\mathcal{UM}_X(H_{G,x_0})$ and $\mathcal{BP}_X(H_{G,x_0})$ respectively.

2.1 UR-matchings in bipartite graphs

By the definition of UR-matchings, the following statements are obviously equivalent for any matching M in a multigraph G:

(i) M is a UR-matching of G;

(ii) M is a UR-matching of the subgraph $G[V(M)]$;

(iii) $|E(C)| > 2|M \cap E(C)|$ holds for any cycle C in G.

For UR-matchings in a bipartite graph, another equivalent statement is given by Golumbic, Hirst and Hedetniemia [8].

Theorem 2.1 ([8]) *$M \in \mathcal{UM}_X(H)$ if and only if $M = \{x_{\pi_i} y_{\tau_i} : i = 1, 2, \cdots, n\}$ for a permutation $\pi_1, \pi_2, \cdots, \pi_n$ of $1, 2, \cdots, n$ and an n-permutation $\tau_1, \tau_2, \cdots, \tau_n$ of $1, 2, \cdots, m$ with $x_{\pi_i} y_{\tau_i} \in E(H)$ for all $i = 1, 2, \cdots, n$ but $x_{\pi_j} y_{\tau_i} \notin E(H)$ for all $1 \le i < j \le n$.*

Theorem 2.1 can be stated equivalently as follows.

Corollary 2.1 *For any $M \subseteq E(H)$ with $|M| = n$, $M \in \mathcal{UM}_X(H)$ if and only if $V(M) \cap Y = \{y_{\tau_i} : i = 1, 2, \cdots, n\}$ holds for some n-permutation $\tau_1, \tau_2, \cdots, \tau_n$ of $1, 2, \cdots, m$ such that y_{τ_i} is a leaf in the subgraph $H - \bigcup_{1 \le s < i} N_H[y_{\tau_s}]$ for all $i = 1, 2, \cdots, n$.*

Corollary 2.1 implies a necessary condition for $\mathcal{UM}_X(H)$ to be non-empty. Let $L(H)$ denote the set of leaves in H.

Corollary 2.2 *If $\mathcal{UM}_X(H) \neq \emptyset$, then $L(H_i) \cap Y \neq \emptyset$ for each component H_i of H.*

But Corollary 2.2 is not true for a non-bipartite graph which contains perfect UR-matchings. An example from [8] is shown in Figure 2, where the graph is non-bipartite and has a perfect UR-matching $\{e_1, e_2, e_3\}$. But it does not have any leaf.

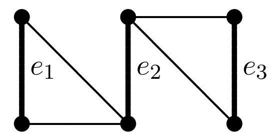


Figure 2: A non-bipartite graph with a perfect UR-matching $\{e_1, e_2, e_3\}$

Hall's Theorem [10] on bipartite graphs is an important result of characterizing bipartite graphs with matchings saturating all vertices in one partite set. By Theorem 2.1, we can get a characterization for $\mathcal{UM}_X(H)$ to be non-empty in terms of the sizes of sets $N_H(S)$, where $S \subseteq X$.

Corollary 2.3 *$\mathcal{UM}_X(H) \neq \emptyset$ if and only if there exists a permutation $\pi_1, \pi_2, \cdots, \pi_n$ of $1, 2, \cdots, n$ such that*

$$|N_H(X_1)| > |N_H(X_2)| > \cdots > |N_H(X_n)| > 0,$$

where $X_i = \{x_{\pi_j} : i \leq j \leq n\}$ and $N_H(X_i) = \bigcup_{x \in X_i} N_H(x)$.

By Corollary 2.3 or Theorem 2.1, if $\mathcal{UM}_X(H) \neq \emptyset$, then H contains at least one leaf $y' \in Y$ in H. We are now going to show that if $\mathcal{UM}_X(H) \neq \emptyset$, then each leaf $y' \in Y$ of H is contained in $V(M)$ for some $M \in \mathcal{UM}_X(H)$.

Proposition 2.1 *Assume that $y' \in Y$ is a leaf of H with $N_H(y') = \{x'\}$. Let $X' = X - \{x'\}$, $H' = H - y'$ and $H'' = H - \{x', y'\}$. The following statement hold:*

(i) *for any $M \in \mathcal{UM}_X(H)$, if $y' \notin V(M)$, then $M \in \mathcal{UM}_X(H')$; otherwise, $M - \{x'y'\} \in \mathcal{UM}_{X'}(H'')$;*

(ii) *if $\mathcal{UM}_X(H) \neq \emptyset$, then $y' \in V(M)$ for some $M \in \mathcal{UM}_X(H)$.*

Proof. (i) follows from Theorem 2.1 directly.

(ii) Assume that $M \in \mathcal{UM}_X(H)$ with $y' \notin V(M)$. By Theorem 2.1, there exist a permutation $\pi_1, \pi_2, \cdots, \pi_n$ of $1, 2, \cdots, n$ and an n-permutation $\tau_1, \tau_2, \cdots, \tau_n$ of $1, 2, \cdots, m$ such that $M = \{x_{\pi_i} y_{\tau_i} : i = 1, 2, \cdots, n\}$ and $x_{\pi_i} y_{\tau_j} \notin E(H)$ for all $1 \leq j < i \leq n$.

Assume that $y' = y_q$ and $x' = x_{\pi_k}$. Then $\tau_i \neq q$ for all $i = 1, 2, \cdots, n$. Let $\gamma_k = q$ and $\gamma_i = \tau_i$ for all i with $1 \leq i \leq n$ and $i \neq k$. Then $\pi_1, \pi_2, \cdots, \pi_n$ is a permutation of $1, 2, \cdots, n$ and $\gamma_1, \gamma_2, \cdots, \gamma_n$ is an n-permutation of $1, 2, \cdots, m$ such that $x_{\pi_i} y_{\gamma_i} \in E(H)$ for all $i = 1, 2, \cdots, n$ but $x_{\pi_i} y_{\gamma_j} \notin E(H)$ for all $1 \leq j < i \leq n$. By Theorem 2.1, $M' = \{x_{\pi_i} y_{\gamma_i}, i = 1, 2, \cdots, n\}$ is a member in $\mathcal{UM}_X(H)$ with $y' = y_q = y_{\gamma_k} \in V(M')$.

Hence (ii) holds. □

2.2 B-parking functions

A characterization of B-parking functions is given below.

Proposition 2.2 *For any mapping $f : X \to \mathbb{N}_0$, $f \in \mathcal{BP}_X(H)$ if and only if there is a permutation $\pi_1, \pi_2, \cdots, \pi_n$ of $1, 2, \cdots, n$ such that for each $i = 1, 2, \cdots, n$, x_{π_i} has at least $f(x_{\pi_i}) + 1$ neighbors which are leaves in the subgraph of H induced by $\bigcup_{i \leq j \leq n} N[x_{\pi_j}]$.*

Proof. (⇒) Assume that $f \in \mathcal{BP}_X(H)$. By the definition of B-parking functions, there exists a vertex $x_{\pi_1} \in X$ such that $|N_H(x_{\pi_1}) \cap L(H)| \geq f(x_{\pi_1}) + 1$.

Assume that $\pi_1, \pi_2, \cdots, \pi_s$ is a s-permutation of $1, 2, \cdots, n$, where $1 \leq s < n$, such that for all $i = 1, 2, \cdots, s$, $|N_H(x_{\pi_i}) \cap L(H[N[X_i]])| \geq f(x_{\pi_i}) + 1$, where $X_i = X - \{x_{\pi_r} : 1 \leq r < i\}$. By the definition of B-parking functions again, there exists a vertex, denoted by $x_{\pi_{s+1}}$, in X_{s+1} such that $|N_H(x_{\pi_{s+1}}) \cap L(H[N[X_{s+1}]])| \geq f(x_{\pi_{s+1}}) + 1$. Repeating this process, a permutation $\pi_1, \pi_2, \cdots, \pi_n$ of N_n can be obtained such that $|N_H(x_{\pi_i}) \cap L(H[N[X_i]])| \geq f(x_{\pi_i}) + 1$ for all $i = 1, 2, \cdots, n$. Observe that X_i is the set $\{x_{\pi_r} : i \leq r \leq n\}$. Thus the necessity holds.

(⇐) Now assume that $\pi_1, \pi_2, \cdots, \pi_n$ is a permutation of $1, 2, \cdots, n$ such that for $i = 1, 2, \cdots, n$, $|N_H(x_{\pi_i}) \cap L(H[N[X_i]])| \geq f(x_{\pi_i}) + 1$ holds, where $X_i = \{x_{\pi_r} : i \leq r \leq n\}$. Let X' be an arbitrary non-empty subset of X and s be the minimum integer in N_n such that $x_{\pi_s} \in X'$. By assumption, x_{π_s} has at least $f(x_{\pi_s}) + 1$ neighbors which are leaves in $H[N[X_s]]$. Observe that $X' \subseteq X_s = \{x_{\pi_r} : s \leq r \leq n\}$, implying that for any $y \in N_H(x_{\pi_s})$, if $y \in L(H[N[X_s]])$, then $y \in L(H[N[X']])$. Thus $|N_H(x_{\pi_s}) \cap L(H[N[X']]) \geq f(x_{\pi_s}) + 1$. Hence $f \in \mathcal{BP}_X(H)$. □

By Proposition 2.2, one can prove the following characterization for members in $\mathcal{BP}_X(H)$ by acyclic orientations of H.

Corollary 2.4 *For any $f : X \to \mathbb{N}_0$, $f \in \mathcal{BP}_X(H)$ if and only if there exists an acyclic orientation D of H such that $od_D(y_j) = 1$ holds for all $j = 1, 2, \cdots, m$ and $f(x_i) < id_D(x_i)$ holds for all $i = 1, 2, \cdots, n$, where $od_D(y_j)$ and $id_D(x_i)$ are respectively the out-degree of y_j and the in-degree of x_i in D.*

Let f be a mapping from X to $\mathbb{N}_0$. For any $X' \subseteq X$ and $x' \in X$, let $f|_{X'}$ be the restriction of f to the set X' and let $f_{(x'\downarrow 1)}$ be the mapping defined by $f_{(x'\downarrow 1)}(x') = f(x') - 1$ and $f_{(x'\downarrow 1)}(x) = f(x)$ for all $x \in X - \{x'\}$. By Proposition 2.2, we have the following result.

Corollary 2.5 *Assume that $y' \in Y \cap L(H)$ and $N_H(y') = \{x'\}$. For any mapping f from X to $\mathbb{N}_0$, the following statements hold:*

(i) *if $f(x) = 0$ for all $x \in X$, then $\mathcal{BP}_X(H) \neq \emptyset$ if and only if $f \in \mathcal{BP}_X(H)$;*

(ii) *$f_{(x'\downarrow 1)} \in \mathcal{BP}_X(H - y')$ if and only if $f \in \mathcal{BP}_X(H)$ and $f(x') \geq 1$;*

(iii) *if $f(x') = 0$, then $f|_{X-\{x'\}} \in \mathcal{BP}_{X-\{x'\}}(H - x')$ if and only if $f \in \mathcal{BP}_X(H)$.*

By applying Propositions 2.1 and 2.2, Theorem 2.1 and Corollary 2.5, it can be shown that $\mathcal{UM}_X(H) \neq \emptyset$ if and only if $\mathcal{BP}_X(H) \neq \emptyset$.

Proposition 2.3 *The following statements are equivalent:*

(i) *$L(H) \cap Y \neq \emptyset$ and for each $y \in L(H) \cap Y$, $y \in V(M)$ for some $M \in \mathcal{UM}_X(H)$;*

(ii) *$\mathcal{UM}_X(H) \neq \emptyset$;*

(iii) *there exist a permutation $\pi_1, \pi_2, \cdots, \pi_n$ of $1, 2, \cdots, n$ and an n-permutation $\tau_1, \tau_2, \cdots, \tau_n$ of $1, 2, \cdots, m$ such that $M = \{x_{\pi_i} y_{\tau_i} : i = 1, 2, \cdots, n\}$ and $x_{\pi_i} y_{\pi_j} \notin E(H)$ for all $1 \leq j < i \leq n$;*

(iv) *$f \in \mathcal{BP}_X(H)$, where f is the mapping defined by $f(x) = 0$ for all $x \in X$;*

(v) *$\mathcal{BP}_X(H) \neq \emptyset$.*

Proof. Observe that (i) ⇔ (ii), (ii) ⇔ (iii), (iii) ⇔ (iv) and (iv) ⇔ (v) follow from Proposition 2.1 (ii), Theorem 2.1, Proposition 2.2 and Corollary 2.5 (i) respectively. □

2.3 $\mathcal{UM}_X(H_{G,x_0})$ and $\mathcal{BP}_X(H_{G,x_0})$

We focus on the special bipartite graph H_{G,x_0} in this subsection. Note that H_{G,x_0} has a bipartition (X, Y), where $X = V - \{x_0\}$ and $Y = E$. Each vertex of Y is of degree 1 or 2 in H_{G,x_0}. As G is connected, $L(H_t) \cap Y \neq \emptyset$ for each component H_t of H_{G,x_0}. Also note that y_i and y_j are parallel edges in G if and only if y_i and y_j have the same set of neighbors in H_{G,x_0}. An example of H_{G,x_0} is shown in Figure 3.

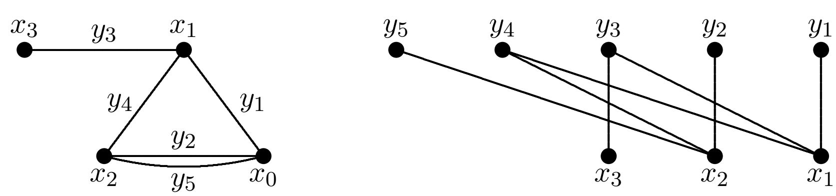
In this subsection, we will show that there is a bijection from $\mathcal{T}(G)$ to $\mathcal{UM}_X(H_{G,x_0})$ and $\mathcal{GP}(G, x_0) = \mathcal{BP}_X(H_{G,x_0})$ holds.

Lemma 2.1 *If G_0 is a disconnected multigraph, then $\mathcal{UM}_X(H_{G_0,x_0}) = \emptyset$.*

Proof. Assume that G_0 is disconnected. Then some component of H_{G_0,x_0} does not have leaves. By Corollary 2.2, $\mathcal{UM}_X(H_{G_0,x_0}) = \emptyset$. □

By Lemma 2.1, we need only to consider connected multigraphs. Let $T \in \mathcal{T}(G)$. Without loss of generality, assume that $E(T) = \{y_i : 1 \leq i \leq n\}$. Recall that M_T denotes the matching $\{x_{\epsilon_i} y_i : i = 1, 2, \cdots, n\}$ of H_{G,x_0}, where x_{ϵ_i} is the end of edge y_i in G such that y_i is contained in the unique path in T from x_0 to x_{ϵ_i}. By the definition of M_T, M_T is characterized by the following lemma.

Lemma 2.2 *$M_T = \{x_{\pi_i} y_{\tau_i} : i = 1, 2, \cdots, n\}$ if and only if $\pi_1, \pi_2, \cdots, \pi_n$ is a permutation of $1, 2, \cdots, n$ such that each y_{τ_i} is an edge in $E_T(V_i, V - V_i)$ incident with x_{π_i}, where $V_i = \{x_0\} \cup \{x_{\pi_j} : 1 \leq j < i\}$ for $i = 1, 2, \cdots, n$.*



(a) G (b) Graph H_{G,x_0}

Figure 3: Graphs G and H_{G,x_0}

Proposition 2.4 *The mapping $\lambda : \mathcal{T}(G) \to \mathcal{UM}_X(H_{G,x_0})$ defined by $\lambda(T) = M_T$ is a bijection.*

Proof. Clearly, if T_1 and T_2 are distinct members in $\mathcal{T}(G)$, then $M_{T_1} \neq M_{T_2}$. Thus, it suffices to prove the following statements:

(i) For any $T \in \mathcal{T}(G)$, $M_T \in \mathcal{UM}_X(H_{G,x_0})$;

(ii) For any $T \in \mathcal{T}(G)$, M_T is the only member in $\mathcal{UM}_X(H_{G,x_0})$ with $V(M_T) \cap Y = E(T)$;

(iii) For any $M \in \mathcal{UM}_X(H_{G,x_0})$, $V(M) \cap Y = E(T)$ holds for some $T \in \mathcal{T}(G)$.

(i) Let $T \in \mathcal{T}(G)$. By Lemma 2.2, $M_T = \{x_{\pi_i} y_{\tau_i} : i = 1, 2, \cdots, n\}$, where $\pi_1, \pi_2, \cdots, \pi_n$ is some permutation of $1, 2, \cdots, n$ such that y_{τ_i} is an edge in $E_T(V_i, V - V_i)$ with x_{π_i} as one end and $V_i = \{x_0\} \cup \{x_{\pi_j} : 1 \leq j < i\}$ for all for $i = 1, 2, \cdots, n$. Thus y_{τ_i} is a leaf in $H_{G,x_0} - \bigcup_{1 \leq j < i} N_{H_{G,x_0}}(y_{\tau_i})$ for all $i = 1, 2, \cdots, n$. As $M_T \subseteq E(H_{G,x_0})$, by Corollary 2.1, $M_T \in \mathcal{UM}_X(H_{G,x_0})$. Thus (i) holds.

(ii) For any $T \in \mathcal{T}(G)$, by definition, H_{T,x_0} is exactly the subgraph $H_{G,x_0}[X \cup E(T)]$. Since $H_{T,x_0} = S(T) - x_0$ has no cycles, $H_{G,x_0}[X \cup E(T)]$ has no cycles, implying that $H_{G,x_0}[X \cup E(T)]$ cannot have two distinct perfect matchings. As each member $M \in \mathcal{UM}_X(H)$ with $V(M) \cap X = E(T)$ is a perfect matching of $H_{G,x_0}[X \cup E(T)]$, (ii) holds.

(iii) Let $M \in \mathcal{UM}_X(H_{G,x_0})$ and $Y' = V(M) \cap Y$. By Corollary 2.1, there is an n-permutation $\tau_1, \tau_2, \cdots, \tau_n$ of $1, 2, \cdots, m$ such that for all $i = 1, 2, \cdots, n$, $y_{\tau_i} \in Y'$ and y_{τ_i} is incident with a unique vertex x_{π_i} in the subgraph $H_{G,x_0} - \bigcup_{1 \leq j < i} N_{H_{G,x_0}}(y_{\tau_i})$, implying that $y_{\tau_i} \in E_G(V_i, V - V_i)$ with x_{π_i} as one end, where $V_i = \{x_0\} \cup \{x_{\pi_j} : 1 \leq j < i\}$. Thus, $G[Y']$ is a tree. Hence (iii) holds. □

Proposition 2.4 shows that the notion of UR-matchings in bipartite graphs is an extension of that of spanning trees in connected multigraphs.

Proposition 2.5 *For any mapping $f : X \to \mathbb{N}_0$, $f \in \mathcal{GP}(G, x_0)$ if and only if $f \in \mathcal{BP}_X(H_{G,x_0})$.*

Proof. Consider the following statements:

(i) $f \in \mathcal{GP}(G, x_0)$;

(ii) for any non-empty subset V' of X, there exists $x_j \in V'$ with $|E_G(x_j, V - V')| > f(x_j)$;

(iii) for any non-empty subset V' of X, there exists $x_j \in V'$ such that x_j has at least $f(x_j)$ neighbors which are leaves in the subgraph of H_{G,x_0} induced by $\bigcup_{x_i \in V'} N_{H_{G,x_0}}(x_i)$;

(iv) $f \in \mathcal{BP}_X(H_{G,x_0})$.

(i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv) follow from the definitions of $\mathcal{GP}(G, x_0)$ and $\mathcal{BP}_X(H_{G,x_0})$ respectively. (ii) $\Leftrightarrow$ (iii) follows from the fact that $y \in E_G(x_j, V - V')$ if and only if y is a vertex in H_{G,x_0} adjacent to x_j and is also a leaf in the subgraph of H_{G,x_0} induced by $\bigcup_{x_i \in V'} N[x_i]$. Hence the result holds. □

A characterization on G-parking functions follows directly from Proposition 2.2 and Proposition 2.5. It was first obtained by Dhar [6].

Corollary 2.6 (Dhar [6]) *For any $f : V - \{x_0\} \to \mathbb{N}_0$, $f \in \mathcal{GP}(G, x_0)$ if and only if there is a permutation $\pi_1, \pi_2, \cdots, \pi_n$ of $1, 2, \cdots, n$ such that $|E_G(x_{\pi_i}, V - V_i)| > f(x_{\pi_i})$ holds for each $i = 1, 2, \cdots, n$, where $V_i = \{x_{\pi_j} : i \le j \le n\}$.*

Applying the notion of acyclic orientations of G, Corollary 2.6 can be equivalently stated as follows.

Corollary 2.7 *For any $f : V - \{x_0\} \to \mathbb{N}_0$, $f \in \mathcal{GP}(G, x_0)$ if and only if there exists an acyclic orientation D of G with x_0 as its unique source such that $f(x_i) < id_D(x_i)$ holds for all $i = 1, 2, \cdots, n$, where $id_D(x_i)$ is the in-degree of x_i in D.*

3 An algorithm

In this section, we design an algorithm, called *Algorithm A*, mainly for the purpose of producing a member f in $\mathcal{BP}_X(H)$ for any $Y' \subseteq Y$ with $\mathcal{UM}_X(H[X \cup Y']) \neq \emptyset$, as stated in Proposition 3.3. By this result, we are able to define a mapping ψ_H from $\mathcal{UM}_X(H)$ to $\mathcal{BP}_X(H)$ which is shown to be a bijection in Theorem 4.1. The outputs of this algorithm are also applied in Section 5 to interpret the member $f \in \mathcal{BP}_X(H)$ which corresponds to any given $M \in \mathcal{UM}_X(H)$ under the mapping ψ_H.

3.1 Algorithm A

The weight function $w : Y \to \mathbb{N}_0$ of H is needed for running Algorithm A. In order to distinguish members in Y, we assume that w is injective and so $w(y_1) \neq w(y_2)$ holds for any two different members $y_1, y_2 \in Y$. The input for Algorithm A below is an order pair (H, Y'), where $Y' \subseteq Y$.

Algorithm A (H, Y'):

A1: Input H with a bipartition (X, Y) and a subset Y' of Y;

A2: Set $i := 1$, $I := X$, $D(x) := \emptyset$ and $F(x) := N_H(x)$ for all $x \in X$;

A3: Set

$$L_I := \{y \in \bigcup_{x \in I} F(x) : y \text{ is a leaf in } H_I\},$$

where H_I is the subgraph of H induced by $I \cup \left(\bigcup_{x\in I} F(x)\right)$. If $L_I = \emptyset$, then output the message "the input does not yield a desired output" and stop;

A4: If $L_I \neq \emptyset$, determine the member y' in L_I with $w(y') < w(y)$ for all $y \in L_I - \{y'\}$ and the unique member $x' \in N_H(y')$;

A5: If $y' \notin Y'$, then set $F(x') := F(x') - \{y'\}$, $D(x') := D(x') \cup \{y'\}$ and go back to Step A3;

A6: If $y' \in Y'$, determine the unique number $\pi_i \in \{1, 2, \cdots, n\}$ and the unique number $\tau_i \in \{1, 2, \cdots, m\}$ such that $x_{\pi_i} = x'$ and $y_{\tau_i} = y'$;

A7: Set $I := I - \{x'\}$. If $|I| > 0$, set $i := i + 1$ and go back to Step A3;

A8: Output π_i, τ_i and $D(x_{\pi_i})$ for all $i = 1, 2, \cdots, n$ and stop.

Running Algorithm A has two possible outcomes. Let $\sigma(H, Y') = 0$ if running Algorithm A with inputs (H, Y') stops with the message "the input does not yield a desired output", and let $\sigma(H, Y') = 1$ otherwise. In the case $\sigma(H, Y') = 1$, running Algorithm A outputs numbers π_i, τ_i and a subset $D(x_{\pi_i})$ of $Y - Y'$ for $i = 1, 2, \cdots, n$, where $\pi_1, \pi_2, \cdots, \pi_n$ is a permutation of $1, 2, \cdots, n$ and $\tau_1, \tau_2, \cdots, \tau_n$ is an n-permutation of $1, 2, \cdots, m$. In this case, π_i, τ_i and $D(x_{\pi_i})$ are rigorously written as $\pi_i(H, Y')$, $\tau_i(H, Y')$ and $D(H, Y', x_{\pi_i})$.

	$i = 1$	$i = 2$	$i = 3$	$i = 4$
$\pi_i(H_1, Y_1)$	4	3	2	1
$\tau_i(H_1, Y_1)$	5	6	1	2
$D(H_1, Y_1, x_{\pi_i})$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Table 1: $\pi_i(H_1, Y_1)$, $\tau_i(H_1, Y_1)$ and $D(H_1, Y_1, x_{\pi_i})$, where $Y_1 = \{y_1, y_2, y_5, y_6\}$

Let's consider some examples. Let H_1 and H_2 be bipartite graphs given in Figure 4 with $w(y_i) = i$. It is not difficult to verify that $\sigma(H_2, Y') = 0$ for all subsets Y' of $\{y_1, y_2, y_3, y_4, y_5\}$. For graph H_1, we have $\sigma(H_1, Y') = 0$ if $Y' = \{y_1, y_2, y_3, y_4\}$. But $\sigma(H_1, Y_i) = 1$ for $i = 1, 2$, where $Y_1 = \{y_1, y_2, y_5, y_6\}$ and $Y_2 = \{y_3, y_4, y_5, y_6\}$, and the outputs are shown in Tables 1 and 2 respectively.

3.2 When does the case "$\sigma(H, Y') = 1$" happen

In this subsection, we shall know when the case "$\sigma(H, Y') = 1$" happens, and how the outputs π_i, τ_i and $D(x_{\pi_i})$ are determined when it happens.

	$i=1$	$i=2$	$i=3$	$i=4$
$\pi_i(H_1,Y_2)$	4	3	1	2
$\tau_i(H_1,Y_2)$	5	6	4	3
$D(H_1,Y_2,x_{\pi_i})$	$\emptyset$	$\emptyset$	$\{y_2\}$	$\{y_1\}$

Table 2: $\pi_i(H_1,Y_2)$, $\tau_i(H_1,Y_2)$ and $D(H_1,Y_2,x_{\pi_i})$, where $Y_2=\{y_3,y_4,y_5,y_6\}$

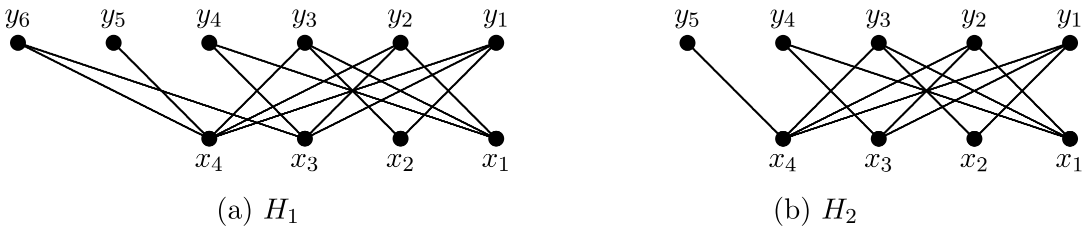


Figure 4: Bipartite graphs H_1 and H_2

If $L(H)\cap Y=\emptyset$, then $\sigma(H,Y')=0$ clearly. If $L(H)\cap Y\neq\emptyset$, we have the following observations from Algorithm A.

Lemma 3.1 *Assume that $L(H)\cap Y\neq\emptyset$ and y' is the member in $L(H)\cap Y$ such that $w(y')$ is the minimum. Let $Y'\subseteq Y$, $Y''=Y'-\{y'\}$, $H'=H-y'$ and $H''=H-\{x',y'\}$, where x' is the only member in $N_H(y')$. The following observations follow from Algorithm A:*

(i) *if $y'\notin Y'$, then $\sigma(H,Y')=\sigma(H',Y')$, and $\sigma(H,Y')=\sigma(H'',Y'')$ otherwise;*

(ii) *if $y'\notin Y'$ and $\sigma(H,Y')=1$, then $\pi_i(H,Y')=\pi_i(H',Y')$ and $\tau_i(H,Y')=\tau_i(H',Y')$ for all $i=1,2,\cdots,n$, and $D(H,Y',x')=D(H',Y',x')\cup\{y'\}$ and $D(H,Y',x)=D(H',Y',x)$ for all $x\in X-\{x'\}$.*

(iii) *if $y'\in Y'$ and $\sigma(H,Y')=1$, then $\pi_1=\pi_1(H,Y')$ and $\tau_1=\tau_1(H,Y')$ such that $y_{\tau_1}=y'$ and $x_{\pi_1}=x'$, and $\pi_i(H,Y')=\pi_{i-1}(H'',Y'')$ and $\tau_i(H,Y')=\tau_{i-1}(H'',Y'')$ for all $i=2,3,\cdots,n$, and $D(H,Y',x')=\emptyset$ and $D(H,Y',x)=D(H'',Y'',x)$ for all $x\in X-\{x'\}$.*

Lemma 3.1 implies that when $\sigma(H,Y')=1$, the outputs π_i and τ_i are independent of the vertices in $Y-Y'$, but each set $D(x_{\pi_i})$ is a subset of $Y-Y'$. Now we are going to show that when $\sigma(H,Y')=1$, the outputs of running Algorithm A can be determined by the following result.

Proposition 3.1 *Let $Y'\subseteq Y$ with $\sigma(H,Y')=1$. Then π_i,τ_i and $D(x_{\pi_i})$ for $i=1,2,\cdots,n$ can be determined by the following statements:*

(i) *for* $i = 1, 2, \cdots, n$, $x_{\pi_i} y_{\tau_i} \in E(H)$ *and* y_{τ_i} *is the member in* $Y' \cap L(H_i)$ *with the minimum weight* $w(y_{\tau_i})$, *where* H_i *denotes the subgraph of* H *induced by* $\bigcup_{i \le s \le n} N[x_{\pi_s}]$;

(ii) *for* $i = 1, 2, \cdots, n$, $D(x_{\pi_i})$ *is the set of those* $y \in (Y - Y') \cap N(x_{\pi_i}) \cap L(H_s)$ *such that* $w(y) < w(y_{\tau_s})$ *holds for some* s *with* $s \le i$.

Proof. (i). By Lemma 3.1 (i), π_i and τ_i for $i = 1, 2, \cdots, n$ are determined by running Algorithm A with input $(H[X \cup Y'], Y')$.

It can be proved by induction on $|X|$. The result is obvious when $|X| = 1$.

Now assume that $|X| \ge 2$. By Lemma 3.1, τ_1 is determined by the fact that y_{τ_1} is the member in $Y' \cap L(H_1)$ with the minimum weight $w(y_{\tau_1})$ and π_1 is determined by the fact that x_{π_1} is the only member in $N_{H_1}(y_{\tau_1})$. By the inductive hypothesis, π_i and τ_i for $i = 2, \cdots, n$ are determined by running Algorithm A with the input $(H', Y' - \{y_{\tau_1}\})$, where $H' = H[(X \cup Y') - \{x_{\pi_1}, y_{\tau_1}\}]$. Thus (i) holds.

(ii) By Lemma 3.1, $\bigcup_{1 \le i \le n} D(x_{\pi_i})$ consists of those $y \in (Y - Y') \cap L(H_s)$ with $w(y) < w(y_{\tau_s})$ for some $s : 1 \le s \le n$. Furthermore, if $y \in (Y - Y') \cap L(H_s)$ with $w(y) < w(y_{\tau_s})$ for some $s : 1 \le s \le n$, then $y \in D(x_{\pi_i})$, where x_{π_i} is the only member in $N_{H_s}(y)$. Clearly $i \ge s$ and x_{π_i} is the only vertex in the set $\{x_{\pi_j} : s \le j \le n\}$ which is adjacent to y.

Hence (ii) holds. □

By Theorem 2.1 and Proposition 3.1, we have the following corollary.

Corollary 3.1 *Let* $Y' \subseteq Y$ *with* $\sigma(H, Y') = 1$. *Then*

(i) $\{x_{\pi_i} y_{\tau_i} : i = 1, 2, \cdots, n\}$ *is a member in* $\mathcal{UM}_X(H)$;

(ii) $x_{\pi_j} y_{\tau_i} \notin E(H)$ *for all* j *with* $j > i$;

(iii) *if* $y_{\tau_i}, y_{\tau_j} \in L(H_r)$, *where* $r \le \min\{i, j\}$, *then* $w(y_{\tau_i}) < w(y_{\tau_j})$ *if and only if* $i < j$, *where* H_r *is the subgraph of* H *induced by* $\bigcup_{r \le s \le n} N[x_{\pi_s}]$.

When $\sigma(H, Y') = 1$, let $M_{H,Y'}$ denote the subset $\{x_{\pi_i} y_{\tau_i} : i = 1, 2, \cdots, n\}$ of $E(H)$. It will be shown in Corollary 3.2 that for any $T \in \mathcal{T}(G)$, if $Y' = E(T)$ and H is the graph H_{G,x_0}, then $M_{H,Y'} = M_T$.

By Corollary 3.1 (i), $M_{H,Y'} \in \mathcal{UM}_X(H)$. Thus $\sigma(H, Y') = 1$ implies that $V(M) \cap Y \subseteq Y'$ holds for some $M \in \mathcal{UM}_X(H)$. Now we show that its converse statement also holds.

Proposition 3.2 *Assume that $Y' \subseteq Y$. Then $\sigma(H, Y') = 1$ if and only if $V(M) \cap Y \subseteq Y'$ holds for some $M \in \mathcal{UM}_X(H)$.*

Proof. By Corollary 3.1 (i), the necessity holds. It suffices to prove the sufficiency.

When $|X| = |Y| = 1$, it is clear that the sufficiency holds. Assume that the sufficiency holds when $2 \le |X| + |Y| < r$. Now consider the case that $|X| + |Y| = r$ and assume that there exists $M \in \mathcal{UM}_X(H)$ with $V(M) \cap Y \subseteq Y'$.

As $\mathcal{UM}_X(H) \neq \emptyset$, by Theorem 2.1, $L(H) \cap Y \neq \emptyset$. Let y' be the member in $L(H) \cap Y$ such that $w(y')$ is the minimum. If $y' \notin Y'$, then $M \in \mathcal{UM}_X(H')$ with $V(M) \cap (Y - \{y'\}) \subseteq Y'$, where $H' = H - y'$, and by the inductive hypothesis, $\sigma(H', Y') = 1$ holds. If $y' \in Y'$, then $M - \{x'y'\} \in \mathcal{UM}_X(H'')$, where $H'' = H - \{x', y'\}$ and x' is the only member in $N_H(y')$, and by the inductive hypothesis, $\sigma(H'', Y'') = 1$ holds, where $Y'' = Y' - \{y'\}$. In both cases, Lemma 3.1 implies that $\sigma(H, Y'') = 1$.

Hence the sufficiency holds. □

3.3 A member of $\mathcal{BP}_X(H)$ when $\sigma(H, Y') = 1$

When $\sigma(H, Y') = 1$, a special member of $\mathcal{BP}_X(H)$ can be determined by the sets $D(H, Y', x)$'s.

Proposition 3.3 *For any $Y' \subseteq Y$ with $\sigma(H, Y') = 1$, the function $f : X \to \mathbb{N}_0$ determined by $f(x) = |D(H, Y', x)|$ for all $x \in X$ is a member in $\mathcal{BP}_X(H)$.*

Proof. We prove it by induction on $|X| + |Y|$. The result is obvious when $|X| = |Y| = 1$ by Proposition 2.2. Assume that the result holds when $2 \le |X| + |Y| < r$. Now consider the case that $|X| + |Y| = r$.

As $\sigma(H, Y') = 1$, $L(H) \cap Y \neq \emptyset$. Let y' be the member in $L(H) \cap Y$ such that $w(y')$ is the minimum. Let x' be the only member in $N_H(y')$.

First consider the case that $y' \notin Y'$. By the inductive hypothesis, the function $g : X \to \mathbb{N}_0$ defined by $g(x) = |D(H', Y', x)|$ for all $x \in X$ is a member in $\mathcal{BP}_X(H')$, where $H' = H - y'$. By Corollary 2.5(ii), the function $f : X \to \mathbb{N}_0$ defined by $f(x') = g(x') + 1$ and $f(x) = g(x)$ for all $x \in X - \{x'\}$ is a member in $\mathcal{BP}_X(H)$. By Lemma 3.1(i), $f(x) = |D(H, Y', x)|$ for all $x \in X$. Thus the result holds in this case.

Now consider the case that $y' \in Y'$. Then $\sigma(H'', Y'') = \sigma(H, Y') = 1$ by Lemma 3.1 (ii), where $Y'' = Y' - \{y'\}$ and $H'' = H - \{x', y'\}$. By the inductive hypothesis, the function $g : X - \{x'\} \to \mathbb{N}_0$ defined by $g(x) = |D(H'', Y'', x)|$ for all $x \in X - \{x'\}$ is a member in $\mathcal{BP}_{X'}(H'')$, where $X' = X - \{x'\}$. By Corollary 2.5(iii), the function $f : X \to \mathbb{N}_0$ defined by $f(x') = 0$ and $f(x) = g(x)$ for all $x \in X - \{x'\}$ is a member in $\mathcal{BP}_X(H)$. By Lemma 3.1(ii), $f(x) = |D(H, Y', x)|$ for all $x \in X$. Thus the result also holds in this case.

Hence the result holds. □

3.4 Outputs of running Algorithm A for H_{G,x_0}

In the next two subsections, we will consider the special bipartite graph H_{G,x_0}.

Note that the weight function $w : E \to \mathbb{N}_0$ for edges of G is also the weight function for members of Y in H_{G,x_0}, which is used in running Algorithm A with input (H_{G,x_0}, Y'), where $Y' \subseteq Y = E$. If $\sigma(H_{G,x_0}, Y') = 1$, simply write $\pi_i = \pi_i(H_{G,x_0}, Y')$, $\tau_i = \tau_i(H_{G,x_0}, Y')$ and $D(x_\pi) = D(H_{G,x_0}, Y', x_\pi)$ for $i = 1, 2, \cdots, n$.

The next result follows from Propositions 3.1 and 3.2.

Proposition 3.4 *$\sigma(H_{G,x_0}, Y') = 1$ if and only if $G[Y']$ is a connected and spanning subgraph of G. Furthermore, if $\sigma(H_{G,x_0}, Y') = 1$, then, for $i = 1, 2, \cdots, n$,*

- (i) *y_{τ_i} is the edge in $Y' \cap E_G(V_i, V - V_i)$ with $w(y_{\tau_i}) \le w(y')$ for all $y' \in Y' \cap E_G(V_i, V - V_i)$ and x_{π_i} is the vertex in $V - V_i$ incident with y_{τ_i}, where $V_i = \{x_0\} \cup \{x_{\pi_s} : 1 \le s < i\}$;*
- (ii) *$D(x_{\pi_i})$ is the set of those edges $y \in Y - Y'$ incident with x_{π_i} such that $y \in E_G(V_s, V - V_s)$ and $w(y) < w(y_{\tau_s})$ hold for some $s \le i$.*

By Lemma 2.2 and Proposition 3.4 (i), for any $T \in \mathcal{T}(G)$, we have the following relation on M_T and $M_{H_{G,x_0}, Y'}$, where $Y' = E(T)$.

Corollary 3.2 *For any $T \in \mathcal{T}(G)$, $M_T = M_{H_{G,x_0}, Y'} = \{x_{\pi_i} y_{\tau_i} : i = 1, 2, \cdots, n\}$, where $Y' = E(T)$.*

For example, let $G = (V, E)$ be the graph shown in Figure 5 (a) and Y' be a subset of E with $G[Y']$ shown in Figure 5 (b), where each number beside an edge e is its weight $w(e)$. As $G[Y']$ is a spanning tree of G, Proposition 3.4 implies that $\sigma(H_{G,x_0}, Y') = 1$.

By Proposition 3.4 (i), $y_{\tau_1}, \cdots, y_{\tau_6}$ are the following edges respectively:

$$x_0x_3, x_3x_4, x_4x_2, x_2x_5, x_2x_1, x_3x_6,$$

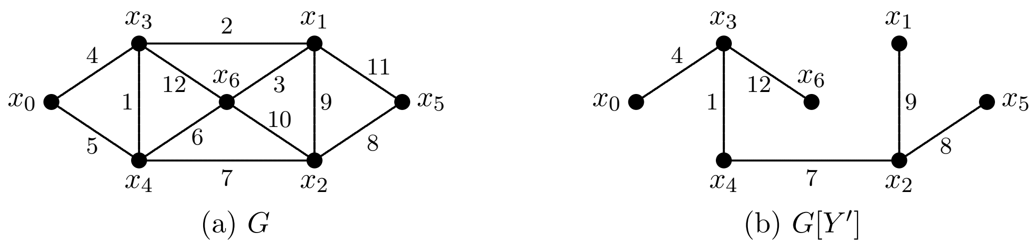


Figure 5: G and a spanning tree $G[Y']$

and $x_{\pi_1}, \cdots, x_{\pi_6}$ are the vertices $x_3, x_4, x_2, x_5, x_1, x_6$ respectively. By applying Proposition 3.4 (ii), we have

$$D(x_3) = D(x_4) = D(x_2) = D(x_5) = \emptyset, D(x_1) = \{x_3x_1\}, D(x_6) = \{x_4x_6, x_2x_6, x_1x_6\}.$$

The next result considers the special case that $Y' = E(T)$ for a given $T \in \mathcal{T}(G)$. It will be applied for proving Theorem 5.1.

Let $P_{i,j}$ denote the unique path in T connecting vertices x_{π_i} and x_{π_j}.

Proposition 3.5 *Let $T \in \mathcal{T}(G)$ and $Y' = E(T)$. Then*

(i) *for $i = 1, 2, \cdots, n$, $G[E_i]$ is a tree with vertex set $\{x_{\pi_s} : 0 \le s \le i\}$, where $E_i = \{y_{\tau_s} : 1 \le s \le i\}$ and $\pi_0 = 0$;*

(ii) *for $i = 1, 2, \cdots, n$, y_{τ_i} is incident with x_{π_i} and is an edge on the path $P_{0,i}$;*

(iii) *if x_{π_i} is a vertex on the path $P_{0,j}$, then $i \le j$ holds;*

(iv) *for any integers $1 \le i, j \le n$, if $\max\{b(y_{\tau_i}), b(y_{\tau_j})\} < \min\{i, j\}$, then $w(y_{\tau_i}) < w(y_{\tau_j})$ if and only if $i < j$, where $b(y_{\tau_j})$ is the number s such that x_{π_s} is the end of y_{τ_j} in G different from x_{π_j}.*

Proof. (i) follows from Proposition 3.4 (i).

(ii) and (iii) follow directly from result (i).

(iv). Let $r = \max\{b(y_{\tau_i}), b(y_{\tau_j})\}$. As $r < \min\{i, j\}$, both y_{τ_i} and y_{τ_j} are members in the set $Y' \cap E_G(V_k, V - V_k)$ for all k with $r < k \le \min\{i, j\}$, where $V_k = \{x_{\pi_t} : k \le t \le n\}$. By Proposition 3.4 (i), $w(y_{\tau_i}) < w(y_{\tau_j})$ if and only if y_{τ_i} is selected before y_{τ_j}, i.e., $i < j$. Thus (iv) holds. □

3.5 The minimum spanning tree

The *minimum spanning tree* of G with respect to w is the spanning tree T_0 of G such that $w(T_0) < w(T)$ holds for all $T \in \mathcal{T}(G) - \{T_0\}$, where $w(T) = \sum_{e\in E(T)} w(e)$. In this subsection, we show that the minimum spanning tree of G is determined by the outputs y_{τ_i}'s of running Algorithm A with input $(H_{G,x_0}, E(G))$. But this property cannot be extended to all bipartite graphs.

Prim's algorithm [19] is a well-known algorithm of determining the minimum spanning tree of a connected multigraph. The way of choosing edges of the minimum spanning tree in G by Prim's algorithm (see [2, 29]) is actually the same as the way of determining edges $y_{\tau_1}, \cdots, y_{\tau_n}$ by Proposition 3.4 (i). Thus the next result follows from Proposition 3.4 (i) and Prim's algorithm.

Corollary 3.3 *For any $Y' \subseteq E$, if $G[Y']$ is connected and spanning, then $E(T_0) = \{y_{\tau_1}, \cdots, y_{\tau_n}\}$ for the minimum spanning tree T_0 of $G[Y']$.*

For any $Y'' \subseteq E$ with $Y' \subset Y''$, when do $G[Y']$ and $G[Y'']$ have the same the minimum spanning tree?

Theorem 3.1 *Let T_0 be the minimum spanning tree of $G[Y']$. For any $Y'' \subseteq E$ with $Y' \subseteq Y''$, T_0 is the minimum spanning tree of $G[Y'']$ if and only if $\left(\bigcup_{1\le i\le n} D(x_{\pi_i})\right) \cap Y'' = \emptyset$.*

Proof. It suffices to show that the two statements below hold:

(a) if $\left(\bigcup_{1\le i\le n} D(x_{\pi_i})\right) \cap Y'' = \emptyset$, then T_0 is the minimum spanning tree of $G[Y'']$;

(b) if $\left(\bigcup_{1\le i\le n} D(x_{\pi_i})\right) \cap Y'' \neq \emptyset$, then T_0 is not the minimum spanning tree of $G[Y'']$.

Assume that $\left(\bigcup_{1\le i\le n} D(x_{\pi_i})\right) \cap Y'' = \emptyset$. By Proposition 3.4 (ii), $\bigcup_{1\le i\le n} D(x_{\pi_i})$ is the set of those edges $y \in Y - Y'$ such that $y \in E_G(V_s, V - V_s)$ and $w(y) \le w(y_{\tau_s})$ hold for some s with $1 \le s \le n$, where $V_s = \{x_{\pi_t} : s \le t \le n\}$. As $\left(\bigcup_{1\le i\le n} D(x_{\pi_i})\right) \cap Y'' = \emptyset$, by Proposition 3.4 (i), y_{τ_i} is the edge in $E_{G[Y'']}(V_i, V - V_i)$ such that $w(y_{\tau_i}) < w(y)$ holds for all edges $y \in E_{G[Y'']}(V_i, V - V_i) - \{y_{\tau_i}\}$ for each $i = 1, 2, \cdots, n$. By Prim's algorithm, $E(T_0) = \{y_{\tau_i} : i = 1, 2, \cdots, n\}$ is the edge set of the minimum spanning tree of $G[Y'']$. Hence (a) holds.

Now consider the case that $\left(\bigcup_{1\le i\le n} D(x_{\pi_i})\right) \cap Y'' \neq \emptyset$. By Corollary 3.3, the edge set of the minimum spanning tree T_0 of $G[Y']$ is $\{y_{\tau_1}, y_{\tau_2}, \cdots, y_{\tau_n}\}$. By Prim's algorithm, the edges of T_0 can be chosen in the order $y_{\tau_1}, y_{\tau_2}, \cdots, y_{\tau_n}$. By the assumption, there exists $y_0 \in \left(\bigcup_{1\le i\le n} D(x_{\pi_i})\right) \cap Y''$. By Proposition 3.4 (ii), $y_0 \in E_{G[Y'']}(V_s, V - V_s)$ and $w(y_0) < w(y_{\tau_s})$ hold for some s with $1 \le s \le n$. By Prim's algorithm again, y_0 is chosen as an edge of the minimum spanning tree of $G[Y'']$ at the step after all edges in $\{y_{\tau_t} : 1 \le t < s\}$ are selected, implying that T_0 is not the minimum spanning tree of $G[Y'']$.

Hence (b) also holds. □

For any $M \in \mathcal{UM}_X(H)$, let $w(M) = \sum_{y \in V(M)\cap Y} w(y)$. A member M_0 in $\mathcal{UM}_X(H)$ is called a *minimum member* in $\mathcal{UM}_X(H)$ if $w(M_0) \le w(M)$ holds for all $M \in \mathcal{UM}_X(H)$. By Corollary 3.3, $\{x_{\pi_i} y_{\tau_i} : i = 1, 2, \cdots, n\}$ is the unique minimum member of $\mathcal{UM}_X(H_{G,x_0})$. However, this result does not hold all bipartite graphs H. An example is shown in Figure 6.

Let H_0 be the bipartite graph shown in Figure 6, where any vertex with an order pair (y_i, w_i) beside is vertex y_i with $w(y_i) = w_i$. Running Algorithm A with input (H_0, Y_0), where $Y_0 = \{y_1, y_2, y_3, y_4\}$, outputs $\pi_i = \tau_i = i$ for $i = 1, 2, 3$. But $M_0 = \{x_i y_i : i = 1, 2, 3\}$ is not the minimum member of $\mathcal{UM}_X(H_0)$, as $M_1 = \{x_2y_2, x_3y_3, x_1y_4\} \in \mathcal{UM}_X(H_0)$ and

$$w(M_1) = w(y_2) + w(y_3) + w(y_4) < w(y_1) + w(y_2) + w(y_3) = w(M_0).$$

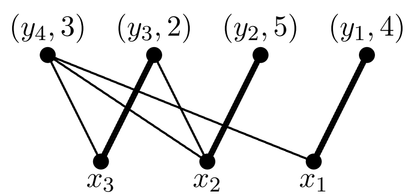


Figure 6: A bipartite graph H_0

Problem 3.1 *For any bipartite graph H with a bipartition (X, Y) and $\mathcal{UM}_X(H) \ne \emptyset$, determine the minimum member of $\mathcal{UM}_X(H)$.*

4 Bijection ψ_H from $\mathcal{UM}_X(H)$ to $\mathcal{BP}_X(H)$

For any $M \in \mathcal{UM}_X(H)$, let $\psi_H(M) = f$, where f is the mapping $f : X \to \mathbb{N}_0$ defined by $f(x_i) = |D(H, Y \cap V(M), x_i)|$ for each $x_i \in X$. By Propositions 3.2 and 3.3, ψ_H is a mapping from $\mathcal{UM}_X(H)$ to $\mathcal{BP}_X(H)$. By its definition, an interpretation of ψ_H is given by Proposition 3.1. We are now going to show that ψ_H is a bijection.

Theorem 4.1 *The mapping $\psi_H : \mathcal{UM}_X(H) \to \mathcal{BP}_X(H)$ defined above is a bijection from $\mathcal{UM}_X(H)$ to $\mathcal{BP}_X(H)$.*

Proof. We first prove that ψ_H is injective by induction on $|X|+|Y|$. When $|X|=|Y|=1$, the conclusion is obvious, as $\mathcal{UM}_X(H)$ has at most one member. Assume that it holds when $|X|+|Y|<k$, where $k\geq 3$. Now consider the case that $|X|+|Y|=k$.

Assume that $\mathcal{UM}_X(H)\neq\emptyset$. By Theorem 2.1, $Y\cap L(H)\neq\emptyset$. Assume that y' is the member in $Y\cap L(H)$ such that $w(y')$ is the minimum. Let x' be the only member in $N_H(y')$.

Let M_1 and M_2 be distinct members in $\mathcal{UM}_X(H)$ and $Y_i=V(M_i)\cap Y$ for $i=1,2$. If $Y_1=Y_2$, then $V(M_1)=V(M_2)$, implying that $M_1=M_2$ by the definition of UR-matchings. Thus $Y_1\neq Y_2$. Let $f_i(x)=|D(H,Y_i,x)|$ for $i=1,2$ and all $x\in X$. We shall show that $f_1\neq f_2$ in the three cases below.

Case 1: $y'\in Y_1-Y_2$ or $y'\in Y_2-Y_1$.

Assume that $y'\in Y_1-Y_2$. By Lemma 3.1, $D(H,Y_1,x')=\emptyset$ while $y'\in D(H,Y_2,x')$. Thus $f_1(x')<f_2(x')$.

Case 2: $y'\notin Y_1\cup Y_2$.

In this case, $M_i\in\mathcal{UM}_X(H')$ for $i=1,2$, where $H'=H-y'$. By the inductive hypothesis, $\psi_{H'}$ is an injective mapping from $\mathcal{UM}_X(H')$ to $\mathcal{BP}_X(H')$, implying that $|D(H',Y_1,x)|\neq|D(H',Y_2,x)|$ for some $x\in X$. By Lemma 3.1(i), for each $i=1,2$, $D(H,Y_i,x')=D(H',Y_i,x')\cup\{y'\}$ and $D(H,Y_i,x)=D(H',Y_i,x)$ for all $x\in X-\{x'\}$, implying that $|D(H,Y_1,x)|\neq|D(H,Y_2,x)|$ for some $x\in X$, i.e., $f_1\neq f_2$.

Case 3: $y'\in Y_1\cap Y_2$.

By Lemma 3.1(ii), for $i=1,2$, $D(H,Y_i,x')=\emptyset$ and $D(H,Y_i,x)=D(H'',Y_i',x)$ for all $x\in X'=X-\{x'\}$, where $H''=H-\{x',y'\}$ and $Y_i'=Y_i-\{y'\}$. Note that $Y_i'=Y\cap V(M_i')$ for $i=1,2$, where $M_i'=M_i-\{x'y'\}$. As $M_1\neq M_2$, we have $M_1'\neq M_2'$. By the inductive hypothesis, $|D(H'',Y_1',x)|\neq|D(H'',Y_2',x)|$ for some $x\in X'$, implying that $|D(H,Y_1,x)|\neq|D(H,Y_2,x)|$. Thus, $f_1\neq f_2$ in this case.

Therefore ψ_H is injective.

It remains to prove that ψ_H is surjective, i.e., the following statement "for any $f\in\mathcal{BP}_X(H)$, there exists $M\in\mathcal{UM}_X(H)$ with $\psi_H(M)=f$" holds. We prove this statement by induction on the value of $|X|+|Y|+\sum_{x\in X}f(x)$, where $f\in\mathcal{BP}_X(H)$. Observe that $|X|+|Y|+\sum_{x\in X}f(x)\geq 2$. When $|X|+|Y|+\sum_{x\in X}f(x)=2$, we have $|X|=|Y|=1$ and $f(x)=0$ for the only member $x\in X$, implying that $H\cong K_2$ and $\psi_H(M)=f$ holds, where $M=E(H)$.

Assume that the above statement holds for any bipartite graph H' with a bipartition (X', Y') and any $f' \in \mathcal{BP}_{X'}(H')$ such that $|X'| + |Y'| + \sum_{x \in X'} f'(x) < r$, where $r \geq 3$. Now we suppose that H is a bipartite graph with a bipartition (X, Y) and $f \in \mathcal{BP}_X(H)$ such that $|X| + |Y| + \sum_{x \in X} f(x) = r$.

As $\mathcal{BP}_X(H) \neq \emptyset$, by Proposition 2.3, we have $\mathcal{UM}_X(H) \neq \emptyset$ and $Y \cap L(H) \neq \emptyset$. Assume that y' is the member in $Y \cap L(H)$ such that $w(y')$ is the minimum and x' is the only member in $N_H(y')$. We shall prove in the two cases below that $\psi_H(M) = f$ holds for some $M \in \mathcal{UM}_X(H)$.

Case 1': $f(x') = 0$.

Let $H'' = H - \{x', y'\}$ and $g = f|_{X'}$, where $X' = X - \{x'\}$. By Corollary 2.5(iii), $g \in \mathcal{BP}_{X'}(H'')$. By the inductive hypothesis, there exists $M' \in \mathcal{UM}_{X'}(H'')$ such that $\psi_{H''}(M') = g$, i.e., $g(x) = |D(H'', V(M') \cap Y, x)|$ for all $x \in X'$. It is clear that $M = M' \cup \{x'y'\} \in \mathcal{UM}_X(H)$. By Lemma 3.1(ii), $D(H, Y', x') = \emptyset$ and $D(H, Y', x) = D(H'', Y'', x)$ for all $x \in X - \{x'\}$, where $Y'' = V(M') \cap Y$ and $Y' = Y'' \cup \{y'\} = V(M) \cap Y$. Thus $f(x') = 0 = |D(H, Y', x')|$ and $f(x) = g(x) = |D(H'', Y'', x)| = |D(H, Y', x)|$ for all $x \in X - \{x'\}$, implying that $\psi_H(M) = f$.

Case 2': $f(x') > 0$.

Let $H' = H - \{y'\}$ and $g = f_{(x' \downarrow 1)}$. By Corollary 2.5(ii), $g \in \mathcal{BP}_X(H')$. By the inductive hypothesis, there exists $M \in \mathcal{UM}_X(H')$ such that $\psi_{H'}(M) = g$, i.e., $g(x) = |D(H', Y', x)|$ for all $x \in X$, where $Y' = V(M) \cap Y$. By Lemma 3.1(i), $D(H, Y', x') = D(H', Y', x') \cup \{y'\}$ and $D(H, Y', x) = D(H', Y', x)$ for all $x \in X - \{x'\}$. Thus $f(x') = g(x') + 1 = |D(H', Y', x')| + 1 = |D(H, Y', x')|$ and $f(x) = g(x) = |D(H', Y', x)| = |D(H, Y', x)|$ for all $x \in X - \{x'\}$, implying that $\psi_H(M) = f$. □

For any $T \in \mathcal{T}(G)$, define $\phi_G(T) = \psi_{H_{G,x_0}}(M_T)$. By Theorem 4.1, Corollary 3.2 and Proposition 2.5, ϕ_G is a bijection from $\mathcal{T}(G)$ to $\mathcal{GP}(G, x_0)$. By Proposition 3.4, ϕ_G can be interpreted by the following result, which first appeared in [5].

Corollary 4.1 *Let $T \in \mathcal{T}(G)$. Assume that vertices $x_{\pi_1}, x_{\pi_2}, \cdots, x_{\pi_n}$ and egdes $y_{\tau_1}, y_{\tau_2}, \cdots, y_{\tau_n}$ of G are determined by Proposition 3.4 (i), where $Y' = E(T)$. If $f = \phi_G(T)$, then, for $i = 1, 2, \cdots, n$, $f(x_{\pi_i})$ is the number of those edges $y' \in E(G) - E(T)$ incident with x_{π_i} and some x_{π_j}, where $0 \leq j < i$, with $w(y') < \max_{j < s \leq i} w(y_{\tau_s})$.*

For example, if G is the graph shown in Figure 5 (a) and T is the spanning tree in Figure 5 (b), then $\phi_G(T)$ is the mapping $f \in \mathcal{GP}(G, x_0)$ given below:

$$f(x_2) = f(x_3) = f(x_4) = f(x_5) = 0, f(x_1) = 1, f(x_6) = 3.$$

5 Interpret B-parking functions

Theorem 4.1 shows that the mapping $\psi_H : \mathcal{UM}_X(H) \to \mathcal{BP}_X(H)$ defined by $\psi_H(M) = f$ is a bijection, where $f(x) = |D(H, V(M) \cap Y, x)|$ for all $x \in X$. In this section, assume that $M \in \mathcal{UM}_X(H)$ and $Y' = V(M) \cap Y$, unless otherwise stated. Also assume that $\pi_i = \pi_i(H, Y')$, $\tau_i = \tau_i(H, Y')$ and $D(x_{\pi_i}) = D(H, Y', x_{\pi_i})$. In this section, we will give an interpretation for f different from Proposition 3.1 (ii).

In Subsection 5.1, we define a unique path $P_{(H,M)}(y)$ in H for each $y \in Y - Y'$ with respect to M. In Subsection 5.2, we introduce the concept "externally B-active members with respect to M in H" by comparing $w(y)$ with $w(y')$ for all those $y' \in Y$ which are in the path $P_{(H,M)}(y)$. In Subsection 5.3, we show that $\bigcup_{x\in X} D(H, Y', x)$ is exactly the set of those members in $Y - Y'$ which are not externally B-active with respect to M in H. In particular, $D(H, Y', x_{\pi_i})$ is the set of those members y in $((Y - Y') \cap N_H(x_{\pi_i})) - \bigcup_{s>i} N_H(x_{\pi_s})$ which are not externally B-active with respect to M in H, where $Y' = V(M) \cap Y$. Finally, in Subsection 5.4, we introduce a generating function $\Omega(H; x, y, z)$ for the members in $\mathcal{UM}(H)$ with three variables. Particularly, $\Omega(H_{G,x_0}; x, y, 0)$ is the Tutte polynomial $T_G(x, y)$.

5.1 The path $P_{(H,M)}(y)$ for each $y \in Y - Y'$

By the definition of π_i and τ_i for $i = 1, 2, \cdots, n$, we have $Y' = \{y_{\tau_i} : i = 1, 2, \cdots, n\}$ and $M = M_{H,Y'} = \{x_{\pi_i} y_{\tau_i} : i = 1, 2, \cdots, n\}$. For any vertex $y \in Y$ and any integer $j \geq 1$, let $n_j(y) = 0$ if $j > d_H(y)$, and let $n_j(y)$ be the j'th largest integer s such that $x_{\pi_s} \in N(y)$ otherwise. In other words, $n \geq n_1(y) > n_2(y) > \cdots > n_{d_H(y)}(y) > n_j(y) = 0$ for all $j > d_H(y)$ and $N(y) = \{x_{\pi_s} : s \in \{n_1(y), \cdots, n_{d_H(y)}(y)\}\}$.

Clearly $n_1(y_{\tau_i}) = i$ for all $i = 1, 2, \cdots, n$ by Corollary 3.1 (i) and (ii). By Proposition 3.1, $D(H, Y', x_{\pi_i}) \subseteq \{y : Y - Y', n_1(y) = i\}$.

For any $y \in Y - Y'$, let $P_{(H,M)}(y)$ be the following maximal M-alternating path in H with y as one end:

$$P_{(H,M)}(y) : y x_{\pi_{j_1}} y_{\tau_{j_1}} \cdots x_{\pi_{j_t}} y_{\tau_{j_t}}$$

where $j_1 = n_1(y)$, $j_i = n_2(y_{\tau_{j_{i-1}}}) > 0$ for all $i = 2, 3, \cdots, t$ and $n_2(y) < j_t$, as shown in Figure 7. Thus $j_1 > j_2 > \cdots > j_t > n_2(y)$. By the maximality of $P_{(H,M)}(y)$, $n_2(y) \geq n_2(y_{\tau_{j_t}}) \geq 0$. Clearly that the path $P_{(H,M)}(y)$ is unique for each y.

For example, if H is the bipartite graph shown in Figure 8 with $w(y_i) = i$ for all i and $M = \{x_i y_i : i = 1, 2, \cdots, 5\} \in \mathcal{UM}_X(H)$, then $Y' = \{y_i : i = 1, 2, \cdots, 5\}$, $\pi_i = \tau_i = i$ for $i = 1, 2, \cdots, 5$. Note that y_6 is the only vertex in $Y - Y'$. As $n_1(y_6) = 5$, $n_2(y_5) = 4$,

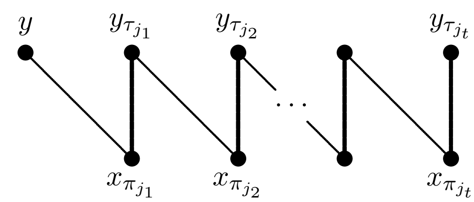


Figure 7: $P_{(H,M)}(y) : yx_{\pi_{j_1}}y_{\tau_{j_1}} \cdots x_{\pi_{j_t}}y_{\tau_{j_t}}$

$n_2(y_4) = 3$ and $n_2(y_3) = 2 = n_2(y_6)$, the path $P_{(H,M)}(y_6)$ is

$$P_{(H,M)}(y_6) : y_6x_5y_5x_4y_4x_3y_3.$$

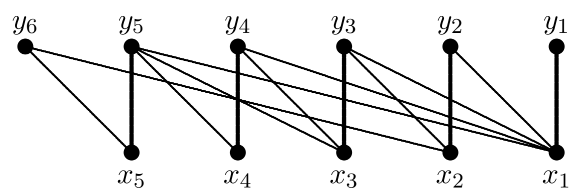


Figure 8: $P_{(H,M)}(y_6) : y_6x_5y_5x_4y_4x_3y_3$ and $n_2(y_6) = n_2(y_3) = 2$

5.2 Externally B-active elements with respect to M

For any $y \in Y - Y'$, y is the only vertex in the path $P_{(H,M)}(y)$ belonging to $Y - Y'$. We say y is *externally B-active with respect to M* in H if $w(y) > w(y_{\tau_{j_r}})$ holds for all $r = 1, 2, \cdots, t$, where $\{y_{\tau_{j_r}} : r = 1, 2, \cdots, t\} = Y' \cap V(P_{(H,M)}(y))$. Let $A_{ex}(H, M)$ denote the set of those members in $Y - Y'$ which are externally B-active with respect to M in H, and let $NA_{ex}(H, M) = (Y - Y') - A_{ex}(H, M)$. Thus $NA_{ex}(H, M)$ is the set of those members in $Y - Y'$ which are not externally B-active with respect to M in H.

Recall that the weight function w of G is a fixed injective mapping from E to $\mathbb{N}_0$. Introduced by Tutte [28], for a given $T \in \mathcal{T}(G)$, an edge y in $E(G) - E(T)$ is said to be *externally active with respect to* T if $w(y) \geq w(y')$ holds for all edges y' in the unique cycle of the subgraph $G[E(T) \cup \{y\}]$, and an edge $y \in E(T)$ is said to be *internally active with respect to* T if $w(y) \geq w(y')$ holds for every edge $y' \in E(G) - E(T)$ with the property that $(E(T) - \{y\}) \cup \{y'\} = E(T')$ holds for some $T' \in \mathcal{T}(G)$. For the definition of these two concepts, the condition "$w(y) \geq w(y')$" can be replaced by "$w(y) \leq w(y')$", as the condition is changed when $w(e)$ is replaced by $K - w(e)$ for each edge e in G, where K is a number in $\mathbb{N}_0$ such that $K - w(e) \geq 0$ for all $e \in E$. Tutte [28] expressed the Tutte polynomial $T_G(x, y)$ as the summation of $x^{ia(T)}y^{ea(T)}$ over all spanning trees T of G, where $ea(T)$ and $ia(T)$ are respectively the number of externally active edges and the number of internally active edges with respect to T.

In the following, we prove that the concept "externally active with respect to T" is extended to the one "externally B-active with respect to M", where $M \in \mathcal{UM}_X(H)$.

Theorem 5.1 *Let $T \in \mathcal{T}(G)$. For any $y \in E(G) - E(T)$, y is externally active respect to T in G if and only if $y \in A_{ex}(H_{G,x_0}, M_T)$.*

Proof. Let $Y' = E(T)$ and let H simply denote H_{G,x_0} in the proof. Thus $\sigma(H, Y') = 1$, and π_i's and τ_i's are determined by Proposition 3.4(i) and have the properties in Proposition 3.5.

Write $x_{\pi_i} \preceq x_{\pi_j}$ if x_{π_i} is a vertex on the path $P_{0,j}$ and $x_{\pi_i} \not\preceq x_{\pi_j}$ otherwise. By Proposition 3.5 (iii), Claim 1 follows directly.

Claim 1: $x_{\pi_i} \preceq x_{\pi_j}$ implies that $i \leq j$.

Thus $x_{\pi_i} \preceq x_{\pi_j}$ if and only if $i \leq j$ and $P_{i,j}$ is part of $P_{0,j}$. In the following, we first compare i and j in the case that $x_{\pi_i} \not\preceq x_{\pi_j}$ and $x_{\pi_j} \not\preceq x_{\pi_i}$. Define $w_{max}(P_{i,j})$ as follows:

$$w_{max}(P_{i,j}) = \begin{cases} -1, & \text{if } E(P_{i,j}) = \emptyset \\ \max\{w(e) : e \in E(P_{i,j})\}, & \text{otherwise.} \end{cases}$$

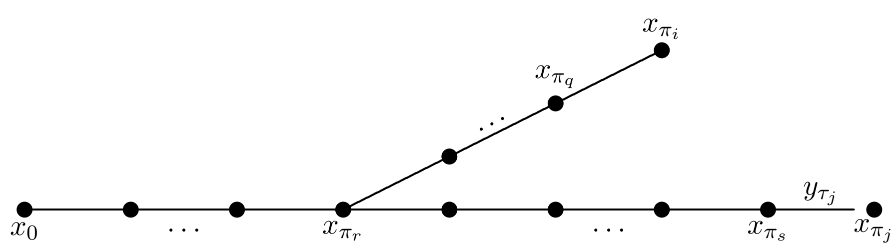


Figure 9: $x_{\pi_r} \preceq x_{\pi_i}$, $x_{\pi_r} \preceq x_{\pi_j}$ but $E(P_{r,i}) \cap E(P_{r,j}) = \emptyset$.

Claim 2: If $x_{\pi_r} \preceq x_{\pi_i}$, $x_{\pi_r} \preceq x_{\pi_j}$ and $E(P_{r,i}) \cap E(P_{r,j}) = \emptyset$, then $w_{max}(P_{r,i}) < w_{max}(P_{r,j})$ implies that $i < j$.

Assume that $w_{max}(P_{r,i}) < w_{max}(P_{r,j})$. We shall prove Claim 2 by induction on the the value of $\rho(i,j) = |E(P_{r,i})| + |E(P_{r,j})|$. By Proposition 3.5 (iv) and the definition of $w_{max}(P_{i,j})$, Claim 2 holds when $|E(P_{r,i})| \leq 1$ and $|E(P_{r,j})| \leq 1$.

Assume that Claim 2 holds when $\rho(i,j) < K$, where $K \geq 3$. Now consider the case that $\rho(i,j) = K$.

Let k be the least possible integer such that y_{τ_k} is an edge on the path $P_{r,j}$ with $w(y_{\tau_k}) > w_{max}(P_{r,i})$. As $w_{max}(P_{r,i}) < w_{max}(P_{r,j})$, such k exists. By Claim 1, $r < k \leq j$. If $k < j$, then $\rho(i,k) < K$ and by the inductive hypothesis, $w_{max}(P_{r,i}) < w(y_{\tau_k}) = w_{max}(P_{r,k})$ implies that $i < k$, and so $i < j$ holds. Thus it suffices to consider the case that $k = j$, i.e., $w_{max}(P_{r,i}) < w(y_{\tau_j})$, but $w_{max}(P_{r,i}) > w(y_{\tau_t})$ for all edges y_{τ_t} on the path $P_{r,j}$ with $t \neq j$.

Let $s = b(y_{\tau_j})$ and $q = b(y_{\tau_i})$, as shown in Figure 9, where $b(y_{\tau_j})$ is defined in Proposition 3.5(iv) (i.e., $b(y_{\tau_j})$ is the number s such that x_{π_s} is the end of y_{τ_j} in G different from x_{π_j}). By Claim 1, $q < i$ and $s < j$. As $\rho(q, j) < K$, by the inductive hypothesis, $w(y_{\tau_j}) > w_{max}(P_{r,i}) \geq w_{max}(P_{r,q})$ implies that $j > q$. As $w_{max}(P_{r,i}) > w_{max}(P_{r,s})$, we have $i > s$ by the inductive hypothesis. Since $b(y_{\tau_j}) = s < i$ and $b(y_{\tau_i}) = q < j$, the inequality $w(y_{\tau_j}) > w_{max}(P_{r,i}) \geq w(y_{\tau_i})$ implies that $j > i$ by Proposition 3.5 (iv).

Hence Claim 2 holds.

Now let y be any edge in $E(G) - E(T)$. Assume that x_{π_i} and $x_{\pi_{j_1}}$ are the two ends of y, where $j_1 > i$, and the unique cycle C in the graph obtained from T by adding y consists of edge y and two edge-disjoint paths $P_{r,i}$ and P_{r,j_1}, where $x_{\pi_r} \preceq x_{\pi_i}$ and $x_{\pi_r} \preceq x_{\pi_{j_1}}$. Thus $r \leq i < j_1$ with the possibility that $i = r$.

Let $x_{\pi_{j_1}} x_{\pi_{j_2}} \cdots x_{\pi_{j_t}}$ be the longest possible subpath of P_{r,j_1} between $x_{\pi_{j_1}}$ and $x_{\pi_{j_t}}$ such that $i < j_t$, as shown in Figure 10. By Claim 1, we have

$$j_1 > j_2 > \cdots > j_t > i \geq b(y_{\tau_{j_t}}), \tag{5.1}$$

where $i = b(y_{\tau_{j_t}})$ if and only if $i = r$ and $b(y_{\tau_{j_t}}) = r$.

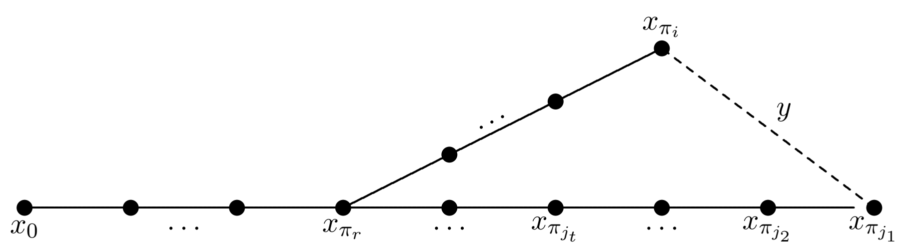


Figure 10: $b(y_{\tau_{j_t}}) \leq i < j_t < \cdots < j_2 < j_1$

As $j_1 > i \geq b(y_{\tau_{j_t}})$, by Claim 2, $w_{max}(P_{r,k}) \leq w_{max}(P_{r,i}) < w_{max}(P_{r,j_t})$, where $k = b(y_{\tau_{j_t}})$, implying that

$$\max\{w_{max}(P_{r,i}), w_{max}(P_{r,j_1})\} = \max\{w(y_{\tau_{j_s}}) : s = 1, 2, \cdots, t\}.$$

Thus the following claim holds.

Claim 3: y is externally active with respect to T in G if and only if $w(y) > w(y_{\tau_{j_s}})$ holds for all $s = 1, 2, \cdots, t$.

On the other hand, by (5.1) and the fact that $n_1(y) = j_1$, $n_2(y_{\tau_{j_s}}) = b(y_{\tau_{j_s}}) = j_{s+1}$ for $s = 1, 2, \cdots, t-1$ and $n_2(y) = b(y) = i \geq k = b(y_{\tau_{j_t}}) = n_2(y_{\tau_{j_t}})$, the path $P_{M_T}(y)$ in H_{G,x_0}

with respect to M_T is exactly the following one:

$$P_{M_T}(y) : yx_{\pi_{j_1}} y_{\tau_{j_1}} \cdots x_{\pi_{j_t}} y_{\tau_{j_t}}.$$

Thus, by definition, the following claim also holds.

Claim 4: $y \in A_{ex}(H_{G,x_0}, M_T)$ if and only if $w(y) > w(y_{\tau_{j_s}})$ holds for all $s = 1, 2, \cdots, t$.

By Claims 3 and 4, the result holds. □

5.3 Interpret B-parking function $f = \psi_H(M)$

By the definition of the path $P_{(H,M)}(y)$, the following lemma follows.

Lemma 5.1 *For any $y \in Y - Y'$, y is adjacent to x_{π_k} on the path $P_{(H,M)}(y)$ if and only if $y \in N_H(x_{\pi_k}) - \bigcup_{k<i\leq n} N_H(x_{\pi_i})$.*

Theorem 5.2 *For any $y \in Y - Y'$ and $1 \leq k \leq n$, $y \in D(H, Y', x_{\pi_k})$ if and only if $y \in N_H(x_{\pi_k}) - \bigcup_{k<i\leq n} N_H(x_{\pi_i})$ and $y \in NA_{ex}(H, M)$.*

Proof. For $i = 1, 2, \cdots, n$, let H_i be the subgraph of H induced by $\sum_{i\leq s\leq n} N[x_{\pi_s}]$. From Algorithm A, $\bigcup_{x\in X} D(H, Y', x)$ is a subset of $Y - Y'$ and can be partitioned into n subsets $D'_1, D'_2, \cdots, D'_n$, where D'_i is the set of those vertices y in H_i having properties below:

(a) $y \in (Y - Y') - \bigcup_{1\leq s<i} D'_i$;

(b) $y \in L(H_i)$;

(c) $w(y) < w(y_{\tau_i})$.

Notice that D'_i is the set of those members $y \in Y - Y'$ which are put into some set $D(x')$, where x' is the only neighbor of y in H_i, at Step A5 in Algorithm A after $y_{\tau_{i-1}}$ is confirmed but before y_{τ_i} is confirmed.

By Corollary 3.1, if $y_{\tau_j} \in L(H_i)$, we have $j \geq i$ and $w(y_{\tau_j}) \geq w(y_{\tau_i})$. Thus the following claim holds:

Claim 1: If $y_{\tau_j} \in L(H_i)$, then $w(y) < w(y_{\tau_j})$ holds for all $y \in D'_i$.

Now let y be a member in $Y - Y'$. Assume that $n_1(y) = j_1$ and the path $P_{(H,M)}(y)$ is $yx_{\pi_{j_1}} y_{\tau_{j_1}} \cdots x_{\pi_{j_t}} y_{\tau_{j_t}}$. By the definition of $P_{(H,M)}(y)$, $j_{s+1} = n_2(y_{\tau_{j_s}})$ for $s = 1, 2, \cdots, t-1$ and $j_1 > j_2 > \cdots > j_t > n_2(y) \geq n_2(y_{\tau_{j_t}})$.

($\Rightarrow$) Assume that $y \in D(H, Y', x_{\pi_k})$. By Proposition 3.1 (ii), $y \in N_H(x_{\pi_k})$. Let i be the minimum integer with $0 < i \leq k$ such that $y \in L(H_i)$ and $w(y) < w(y_{\tau_i})$. Such i exists by Proposition 3.1 (ii). Thus $y \in D'_i$. Clearly, $k = n_1(y) = j_1$ and so x_{π_k} (i.e., $x_{\pi_{j_1}}$) is the vertex on the path $P_{(H,M)}(y)$ adjacent to y. It remains to show that $y \in NA_{ex}(H, M)$.

As $y \in L(H_i)$, we have $q < i \le j_1$, where $q = n_2(y)$. Note that $j_1 > j_2 > \cdots > j_t > n_2(y) = q \ge n_2(y_{\tau_{j_t}})$. Thus $j_{s+1} < i \le j_s$ holds for some s with $1 \le s \le t$, where assume that $j_{t+1} = n_2(y) = q$ when $s = t$. Then $y_{\tau_{j_s}} \in L(H_i)$. By Claim 1, $w(y) < w(y_{\tau_{j_s}})$. By definition, $y \in NA_{ex}(H, M)$.

Hence the necessity holds.

($\Leftarrow$) Now assume that $y \in NA_{ex}(H, M)$. Assume that $j_1 = n_1(y)$. We will show that $y \in D(H, Y', x_{\pi_{j_1}})$.

On the contrary, suppose that $y \notin D(H, Y', x_{\pi_{j_1}})$. By Proposition 3.1 (ii), $y \notin D(H, Y', x_{\pi_s})$ for all $s = 1, 2, \cdots, n$, implying that $y \notin D'_i$ for all $i = 1, 2, \cdots, n$.

As $j_1 = n_1(y)$ and $q = n_2(y)$, $y \in L(H_i)$ for all i with $q < i \le j_1$. For each i with $q < i \le j_1$, as $y \notin D'_i$, we have $w(y) > w(y_{\tau_i})$ by property (c). Particularly, as $q < j_t < \cdots < j_1$, $w(y) > w(y_{\tau_{j_s}})$ holds for all $s = 1, 2, \cdots, t$, implying that y is externally B-active with respect to M in H. Thus $y \notin NA_{ex}(H, M)$, a contradiction.

Hence the sufficiency holds. □

By Theorem 5.2 and the definition of ψ_H, we have the following corollaries.

Corollary 5.1 *Let $M \in \mathcal{UM}_X(H)$. If $f = \psi_H(M)$, then, $f(x_{\pi_i})$ is the size of the set $(N_H(x_{\pi_i}) \cap NA_{ex}(H, M)) - \bigcup_{i<s\le n} N_H(x_{\pi_s})$ for all $i = 1, 2, \cdots, n$.*

Corollary 5.2 *Let $M \in \mathcal{UM}_X(H)$. If $f = \psi_H(M)$, then*

$$\sum_{x\in X} f(x) = |NA_{ex}(H, M)| \le |Y| - |X|.$$

Now we apply Theorem 5.1 to find another interpretation for G-parking functions of G.

Let $T \in \mathcal{T}(G)$. Write $x_{\pi_i} \ll_T x_{\pi_j}$ for all i, j with $0 \le i < j \le n$. For any two vertices x' and x in G, let $P_T(x', x)$ denote the unique path in T between x' and x.

Proposition 5.1 *For any two different vertices x' and x in G, the following statements are equivalent:*

(i) $x' \ll_T x$;

(ii) $w_{max}(P_T(x'', x')) < w_{max}(P_T(x'', x))$, *where* x'' *is the vertex in both paths* $P_T(x_0, x')$ *and* $P_T(x_0, x)$ *with* $E(P_T(x'', x')) \cap E(P_T(x'', x)) = \emptyset$;

(iii) *if* y *is an edge in* $E(G) - E(T)$ *joining* x *and* x', *then* x *is the vertex* x_{π_j} *with* $j = n_1(y)$, *where* $\pi_s = \pi_s(H_{G,x_0}, E(T))$ *for* $s \in \{1, 2, \cdots, n\}$;

(iv) *if* y *is an edge in* $E(G) - E(T)$ *joining* x *and* x', *then* x *is the vertex in the path* $P_{(H,M)}(y)$ *adjacent to* y, *where* $Y' = E(T)$.

Proof. Claims 1 and 2 in the proof of Theorem 5.1 imply that (i) $\Leftrightarrow$ (ii), while the definition of the path $P_{(H,M)}(y)$ implies that (iii) $\Leftrightarrow$ (iv). Finally, by the definition of the ordering $\ll_T$ and the definition of $n_1(y)$, (i) $\Leftrightarrow$ (iii) follows. □

Recall that the mapping $\phi_G : \mathcal{T}(G) \to \mathcal{GP}(G, x_0)$ is defined by $\phi_G(T) = \psi_{H_{G,x_0}}(M_T)$, where $M_T = \{x_{\pi_i} y_{\tau_i} : i = 1, 2, \cdots, n\}$ by Corollary 3.2. By Corollary 5.1 and Proposition 5.1, we get the following interpretation for ϕ_G which is different from the one in Corollary 4.1.

Corollary 5.3 *Let* $T \in \mathcal{T}(G)$. *If* $f = \phi_G(T)$, *then, for any* $x \in V - \{x_0\}$, $f(x)$ *is the number of those edges* $y \in E(G) - E(T)$ *such that* y *is not externally active with respect to* T *in* G *and* y *is incident with* x *and* x', *where* $x' \ll_T x$.

By Corollaries 5.2 and 5.3, we have the following conclusion.

Corollary 5.4 *Let* $T \in \mathcal{T}(G)$. *If* $f = \phi_G(T)$, *then*

$$ea(T) + \sum_{x \in X} f(x) = |E(G)| - |V(G)| + 1,$$

where $ea(T)$ *is the number of externally active edges with respect to* T *in* G.

5.4 A generating function $\Omega(H; x, y, z)$

Let $M \in \mathcal{UM}_X(H)$. For any $x_{\pi_q} \in X$, let $R(x_{\pi_q})$ denote the following unique path:

$$x_{\pi_{j_1}} y_{\tau_{j_1}} x_{\pi_{j_2}} y_{\tau_{j_2}} \cdots x_{\pi_{j_s}} y_{\tau_{j_s}},$$

where $j_1 = q$, $j_{i+1} = n_2(y_{\tau_{j_i}})$ for $i = 1, 2, \cdots, s-1$ and $y_{\tau_{j_s}} \in L(H)$. For any $y' \in Y - V(M)$ and $r \geq 1$, if $t_r = n_r(y') \geq 1$, let $Q_r(y')$ be the path in H formed by combining edge $y' x_{\pi_{t_r}}$ and path $R(x_{\pi_{t_r}})$. In the case that $y' \in L(H)$ (i.e., $n_2(y') < 1$), assume that $Q_2(y')$ consists of vertex y' only. Let $k = 0$ if $V(Q_1(y')) \cap V(Q_2(y')) \cap X = \emptyset$, and let k be the largest integer with $x_{\pi_k} \in V(Q_1(y')) \cap V(Q_2(y')) \cap X$ otherwise. Let $C_H(y')$ be the set $\{y_{\tau_u} \in V(Q_1(y')) \cup V(Q_2(y')) : u > k, y_{\tau_u} \neq y'\}$.

For example, if H is the graph in Figure 8 and $M = \{x_iy_i : i = 1, 2, \cdots, 5\}$, then $Q_1(y_6)$ is the path $y_6x_5y_5x_4y_4x_3y_3x_2y_2x_1y_1$ and $Q_2(y_6)$ is the path $y_6x_2y_2x_1y_1$. Thus $k = 2$ and $C_H(y_6) = \{y_5, y_4, y_3\}$. For the bipartite graph H_{G,x_0} and $M = M_T$, where $T \in \mathcal{T}(G)$, $C_{H_{G,x_0}}(y')$ corresponds to the set of edges $y \neq y'$ in the unique cycle of $G[E(T) \cup \{y'\}]$, where $y' \in E(G) - E(T)$.

For any $y_i \in V(M) \cap Y$, y_i is said to be *internally B-active with respect to M* if $w(y_i) > w(y')$ holds for each $y' \in Y - V(M)$ with $y_i \in C(y')$. Let $A_{in}(H, M)$ be the set of internally B-active members with respect to M in H.

Define a function $\Omega(H; x, y, z)$ with three variable x, y, z as follows:

$$\Omega(H; x, y, z) = \sum_{S \subseteq X} z^{|X|-|S|} \sum_{M \in \mathcal{UM}_S(H)} x^{ia_S(M)} y^{ea_S(M)}, \tag{5.2}$$

where $ia_S(M) = |A_{in}(H[N[S]], M)|$ and $ea_S(M) = |A_{ex}(H[N[S]], M)|$.

If $\Omega(H; 1, 1, z) = \sum_{i \geq 0} c_i z^i$, then c_i is the number of members $M \in \mathcal{UM}(H)$ with $|M| = |X| - i$. In particular, $c_0 = |\mathcal{UM}_X(H)|$.

If $\Omega(H; x, y, 0) = \sum_{i,j \geq 0} u_{i,j} x^i y^j$, then $u_{i,j}$ is the number of members $M \in \mathcal{UM}_X(H)$ with $|A_{in}(H, M)| = i$ and $|A_{ex}(H, M)| = j$ (i.e., $|NA_{ex}(H, M)| = |Y| - |X| - j$).

If $\Omega(H; 1, y, 0) = \sum_{j \geq 0} d_j y^j$, then d_j is the number of members $M \in \mathcal{UM}_X(H)$ with $|A_{ex}(H, M)| = j$, i.e., $|NA_{ex}(H, M)| = |Y| - |X| - j$. By Corollary 5.2, d_j is the number of members f in $\mathcal{BP}_X(H)$ with $\sum_{x \in X} f(x) = |Y| - |X| - j$. By Corollary 5.4, if $H = H_{G,x_0}$, then d_j is the number of members $f \in \mathcal{GP}(G, x_0)$ with $\sum_{x \in X} f(x) = |E(G)| - |V(G)| + 1 - j$.

For any $T \in \mathcal{T}(G)$, an edge $e \in E(T)$ is internally active with respect to T in G if and only if $e \in A_{in}(H, M_T)$. Thus, $\Omega(H_{G,x_0}; x, y, 0)$ is the Tutte polynomial $T_G(x, y)$.

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