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A ZERO-FREE INTERVAL FOR CHROMATIC POLYNOMIALS OF NEARLY 3-CONNECTED PLANE GRAPHS*

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Abstract. Let $G = (V, E)$ be a nonseparable plane graph on n vertices with at least two edges. Suppose that G has outer face C and that every 2-vertex-cut of G contains at least one vertex of C. Let $P_G(q)$ denote the chromatic polynomial of G. We show that $(-1)^n P_G(q) > 0$ for all $1 < q \le 1.2040\ldots$. This result is a corollary of a more general result that $(-1)^n Z_G(q, \mathbf{w}) > 0$ for all $1 < q \le 1.2040\ldots$, where $Z_G(q, \mathbf{w})$ is the multivariate Tutte polynomial of G, $\mathbf{w} = \{w_e\}_{e\in E}$, $w_e = -1$ for all e which are not incident to a vertex of C, $w_e \in W_2$ for all $e \in E(C)$, $w_e \in W_1$ for all other edges e, and W_1, W_2 are suitably chosen intervals with $-1 \in W_1 \subset W_2 \subseteq (-2, 0)$.

Key words. planar graph, Potts model partition function, multivariate Tutte polynomial, chromatic polynomial, zeros

AMS subject classifications. 05C10, 05C15, 05C31

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1. Introduction.

The study of chromatic polynomials of graphs was initiated by Birkhoff [3] for planar graphs in 1912 and, for general graphs, by Whitney [14], [15] in 1932. Inspired by the 4-color conjecture, Birkhoff and Lewis [4] obtained results concerning the distribution of the real zeros of chromatic polynomials of planar graphs and made the stronger conjecture that chromatic polynomials of planar graphs have no real zeros greater than or equal to four. (All graphs considered in this introductory section are assumed to be loopless.) Their hope was that results from analysis and algebra could be used to prove their stronger conjecture and hence deduce that the 4-color conjecture was true. This has not yet occurred: indeed the 4-color conjecture is now a theorem [1], [2], [9], but the stronger conjecture of Birkhoff and Lewis remains unsolved. Nevertheless, many results have been obtained concerning the zero distribution of chromatic polynomials both on the real line and in the complex plane; see the survey articles [6], [10].

We will refer to the zeros of the chromatic polynomial $P_G(q)$ of a graph G as *chromatic roots* of G. It follows from results of Tutte [13, pages 243–266] and this paper's second author [5] that no graph can have a chromatic root in the real intervals $(-\infty, 0)$, $(0, 1)$, or $(1, 32/27]$. Furthermore, these zero-free intervals are maximal: it is easy to see that 0 and 1 are chromatic roots of any graph with at least one edge, and a sequence of (planar) graphs with chromatic roots tending to $32/27$ from above is given in [5]. Thomassen [12] went further by showing that these are the only zero-free intervals for chromatic polynomials: for each $r \in (32/27, \infty)$ and all $\epsilon > 0$ he constructed a graph which has a chromatic root in $(r - \epsilon, r + \epsilon)$.

The graphs used in the constructions of [5], [12] are 2-connected, but contain many 2-vertex-cuts, so it is conceivable that the $(1, 32/27]$ zero-free interval can be extended

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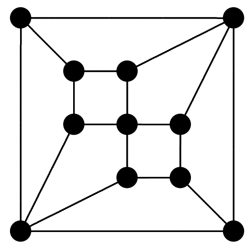

Fig. 1.1. *The Herschel graph.*

for the special family of 3-connected graphs. Indeed, it is conjectured [5, Conjecture 5(a)] that $(1, \alpha)$ is a zero-free interval for chromatic polynomials of 3-connected graphs where $\alpha = 1.781\ldots$ is a chromatic root of $K_{3,4}$. For 3-connected planar graphs the smallest chromatic root we know is $\beta = 1.840\ldots$, which is a chromatic root of the Herschel graph (see Figure 1.1).

The main purpose of this paper is to give some evidence in support of the above-mentioned conjecture by showing that the zero-free interval $(1, 32/27] = (1, 1.185\ldots]$ can at least be extended for 3-connected planar graphs to $(1, 1.2040\ldots]$. Our inductive proof technique requires us to work with the slightly larger family of *nearly* 3*-connected plane graphs*, i.e., nonseparable plane graphs with at least two edges and with the property that every 2-vertex-cut set contains at least one vertex on the outer face. We will show the following result.

Theorem 1.1. *Let G be a nearly* 3*-connected plane graph on n vertices. Then $(-1)^n P_G(q) > 0$ for all $q \in (1, \gamma]$, where $\gamma = (5 - \rho^2)/4 \sim 1.2040\ldots$ and $\rho \sim 0.4289\ldots$ is a root of $x^{10} - 8x^9 + 12x^8 + 40x^7 - 82x^6 - 72x^5 + 68x^4 + 56x^3 + 313x^2 - 16x - 56 = 0$.*

We will also construct a sequence of nearly 3-connected plane graphs with chromatic roots converging to γ from above.

Our proof of Theorem 1.1 uses a weighted generalization of the chromatic polynomial called the multivariate Tutte polynomial (or Potts model partition function); see [11]. It is similar to that of a recent result of Jackson and Sokal [7] which extended the zero-free intervals $(-\infty, 0)$, $(0, 1)$, and $(1, 32/27]$ from chromatic polynomials to multivariate Tutte polynomials. The basic idea is to consider a nearly 3-connected plane graph G with edge weights belonging to three specified intervals (depending on whether they lie on the outer face or, if not, are incident to a vertex of the outer face). If $G - e$ and G/e are both nearly 3-connected for some edge e incident to the outer face, then we use the deletion/contraction lemma and induction. If not, then we show that G has a subgraph H with only two vertices of attachment x, y in G, and with x on the outer face. We replace H by a single edge xy with weight w_H chosen to represent the contribution of H to the multivariate Tutte polynomial of G. (The weight intervals are constructed to be closed under this replacement operation.) We then apply induction to the smaller graph.

An outline of the paper is as follows. Section 2 contains some preliminary graph theoretical results and introduces the multivariate Tutte polynomial. Our proof of the weighted generalization of Theorem 1.1 is given in section 3. We construct the sequence of graphs which have chromatic roots converging to γ in section 4. We close by giving some open problems in section 5.

2. Definitions and preliminary results. All graphs considered henceforth are allowed to have loops and multiple edges unless explicitly stated otherwise. We will refer to graphs with no loops or multiple edges as *simple graphs*. Let G be a graph and k be a nonnegative integer. We say that G is k-connected if $|V(G)| > k$, and $G - U$ is connected for all $U \subseteq V(G)$ with $|U| < k$. A *k-separation of G* is a pair of subgraphs $\{G_1, G_2\}$ such that $G_1 \cup G_2 = G$, $E(G_1) \cap E(G_2) = \emptyset$, $|V(G_1) \cap V(G_2)| = k$, and $|E(G_i)| \geq k$ for each $i = 1,\ 2$. We refer to $V(G_1) \cap V(G_2)$ as a *k-vertex-cut* of G. The graph G is *separable* if it has an h-separation for some $0 \leq h \leq 1$. Otherwise G is said to be *nonseparable*. Thus G is nonseparable if and only if G consists of a single vertex and at most one loop, or G is loopless and is either 2-connected or consists of two vertices joined by at least one edge.

We shall need the following result which follows from a more general theorem of Mader [8, Satz 1].

LEMMA 2.1. *Let G be a 2-connected simple graph and C be a cycle in G. Suppose each vertex of C has degree at least 3. Then $G - e$ is 2-connected for some $e \in E(C)$.*

Let G be a graph and $U \subseteq V(G)$. A *bridge* of U in G is a subgraph B such that B consists of either two vertices of U joined by a single edge of G, or a component H of $G - U$ together with the set of vertices $W \subseteq U$ which are adjacent to H in G and all edges of G joining H to W. We say that B is a *trivial bridge* in the former case and is a *nontrivial bridge* in the latter case. We refer to the vertices of $U \cap V(B)$ as *vertices of attachment* of B on U. When F is a subgraph of G we refer to the bridges of $V(F)$ in $G - E(F)$ as *F-bridges* in G.

2.1. Nearly 3-connected plane graphs. We use the term *plane graph* to mean a planar graph which has been imbedded in the plane. We assume that all minors of a plane graph G are given an imbedding which has been obtained from that of G by deleting and contracting edges. Recall that G is *nearly 3-connected* if it is a nonseparable graph with at least two edges and, for all 2-separations $\{G_1, G_2\}$ of G, $V(G_1) \cap V(G_2)$ contains at least one vertex on the outer face of G. Our inductive proof of Theorem 1.1 will use the following hereditary properties of near 3-connectivity.

LEMMA 2.2. *Let G be a nearly 3-connected plane graph, and let C be the bounding cycle of its outer face.*

(a) *Suppose that $e = uv \in E(C)$. Let $G' \in \{G - e, G/e\}$. If G' is nonseparable, then G' is nearly 3-connected.*

(b) *Suppose that $\{F, H\}$ is a 2-separation of G with $V(F) \cap V(H) = \{y, z\}$ and $C \subseteq H$. Let $F + h$ be obtained from F by adding a new edge $h = yz$ in the outer face of F, and let $G' \in \{F, F + h, (F + h)/h\}$. If G' is nonseparable, then G' is nearly 3-connected.*

Proof. We prove (a) and (b) simultaneously. Let $G' \in \{G - e, G/e, F, F + h, (F + h)/h\}$, and assume that G' is nonseparable. Let C' be the bounding cycle of the outer face of G'. Suppose that G' has a 2-separation $\{G'_1, G'_2\}$ with $V(G'_1) \cap V(G'_2) \cap V(C') = \emptyset$. Relabeling if necessary, we have $V(C') \subseteq V(G'_2) - V(G'_1)$. Let $G_2 = G - (V(G'_1) - V(G'_2)) - E(G'_1)$. Then $\{G'_1, G_2\}$ is a 2-separation of G with $V(C) \subseteq V(G_2) - V(G'_1)$. This contradicts the hypothesis that G is nearly 3-connected. ∎

2.2. The multivariate Tutte polynomial. Our proof of Theorem 1.1 will use the following extension of chromatic polynomials to weighted graphs. Let $G = (V, E)$ be a graph with vertex set V and edge set E. The *multivariate Tutte polynomial* of G is, by definition, the polynomial

$$Z_G(q, \mathbf{w}) = \sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} w_e, \tag{2.1}$$

where q and $\mathbf{w} = \{w_e\}_{e \in E}$ are commuting indeterminates, and $k(A)$ denotes the number of connected components in the subgraph (V, A). We will refer to the pair $(G, \mathbf{w})$ as a *weighted graph*. We use $Z_G(q, w)$ to denote the two-variable polynomial in which all edge weights are equal to w. This polynomial can be transformed to the standard Tutte polynomial by a simple change of variables, and it satisfies $Z_G(q, -1) = P_G(q)$.

The following lemma gives a recursive procedure for calculating $Z_G(q, \mathbf{w})$; see [11].

LEMMA 2.3. (deletion/contraction lemma). *Let $(G, \mathbf{w})$ be a weighted graph and e be an edge of G. Then $Z_G(q, \mathbf{w}) = Z_{G-e}(q, \mathbf{w}|_{G-e}) + w_e Z_{G/e}(q, \mathbf{w}|_{G-e})$.*

2.2.1 Effective weights. Suppose $(G, \mathbf{w})$ is a weighted graph, and F, H are connected subgraphs of G with $F \cup H = G$, $E(F) \cap E(H) = \varnothing$ and $V(F) \cap V(H) = \{u, v\}$. We can calculate the "effective weight" of F in $(G, \mathbf{w})$ as follows. Let $(F + h, \mathbf{w}|_F, -1)$ be obtained from $(F, \mathbf{w}|_F)$ by adding a new edge $h = uv$ to F with weight -1. Let $F_{uv} = (F + h)/h$. Let $(H + e_F, \mathbf{w}|_H, w_F)$ be obtained from $(H, \mathbf{w}|_H)$ by adding an edge $e_F = uv$ with weight w_F.

LEMMA 2.4. *Let*

$$w_F = \frac{qZ_{F_{uv}}(q, \mathbf{w}|_F) - Z_F(q, \mathbf{w}|_F)}{Z_F(q, \mathbf{w}|_F) - Z_{F_{uv}}(q, \mathbf{w}|_F)}. \tag{2.2}$$

Then

$$Z_G(q, \mathbf{w}) = \frac{Z_F(q, \mathbf{w}|_F) - Z_{F_{uv}}(q, \mathbf{w}|_F)}{q(q-1)} Z_{H+e_F}(q, \mathbf{w}|_H, w_F). \tag{2.3}$$

Furthermore, we have

$$Z_G(q, \mathbf{w}) = \frac{Z_{F+h}(q, \mathbf{w}|_F, -1)}{q(q-1)} Z_{H+e_F}(q, \mathbf{w}|_H, w_F) \tag{2.4}$$

and

$$w_F = \frac{(q-1)Z_{F_{uv}}(q, \mathbf{w}|_F)}{Z_{F+h}(q, \mathbf{w}|_F, -1)} - 1, \tag{2.5}$$

$$= \frac{(q-1)Z_{F_{uv}}(q, \mathbf{w}|_F)}{Z_{F+h}(q, \mathbf{w}|_F, -1)} - q. \tag{2.6}$$

Proof. Equation (2.3) follows from [11, Proposition 4.2] (in particular, eq. (4.40) given in the proof of this proposition)]. Equations (2.4), (2.5), and (2.6) are simple reformulations of (2.2) using the fact that

$$Z_{F+h}(q, \mathbf{w}|_F, -1) = Z_F(q, \mathbf{w}|_F) - Z_{F_{uv}}(q, \mathbf{w}|_F). \qquad \square$$

We shall refer to the value of w_F given in the above lemma as the *effective weight of F in $(G, \mathbf{w})$*.

Two special cases of Lemma 2.4 are particularly useful: when F is a cycle of length two (*parallel reduction*) and when F is a path of length two (*series reduction*).

LEMMA 2.5. *Let $(G, \mathbf{w})$ be a weighted graph.*

(a) *Suppose that G has two edges e_1, e_2 with the same end vertices u, v. Let $H = G - \{e_1, e_2\}$, and let $(H + f, \mathbf{w}|_H, w_{e_1} \| w_{e_2})$ be obtained from $(H, \mathbf{w}|_H)$ by adding a new edge $f = uv$ of weight*

$$w_{e_1} \| w_{e_2} = w_{e_1} + w_{e_2} + w_{e_1} w_{e_2}.$$

Then $Z_G(q, \mathbf{w}) = Z_{H+f}(q, \mathbf{w}|_H, w_{e_1} \| w_{e_2})$.

(b) *Suppose G has a vertex x of degree 2 incident with two edges $e_1 = xu$ and $e_2 = xv$. Let $H = G - x$, and let $(H + f, \mathbf{w}|_H, w_{e_1} \bowtie_q w_{e_2})$ be obtained from $(H, \mathbf{w}|_H)$ by adding a new edge $f = uv$ of weight*

$$w_{e_1} \bowtie_q w_{e_2} = \frac{w_{e_1} w_{e_2}}{q + w_{e_1} + w_{e_2}}.$$

Then $Z_G(q, \mathbf{w}) = (q + w_{e_1} + w_{e_2}) Z_{H+f}(q, \mathbf{w}|_H, w_{e_1} \bowtie_q w_{e_2})$.

The following properties of parallel and series reduction can be verified by elementary calculus.

LEMMA 2.6. *Let z_1, q be fixed real numbers with $q > 0$.*

(a) *For all $z \in (-\infty, \infty)$: $z_1 \| z = -1$ when $z_1 = -1$, $z_1 \| z$ is a strictly increasing continuous function of z when $z_1 > -1$, and $z_1 \| z$ is a strictly decreasing continuous function of z when $z_1 < -1$.*

(b) *For all $z \in (-\infty, -q - z_1) \cup (-q - z_1, \infty)$: $z_1 \bowtie_q z = -q$ when $z_1 = -q$, $z_1 \bowtie_q z$ is a strictly increasing continuous function of z when $z_1 \in (-\infty, -q) \cup (0, \infty)$, and $z_1 \bowtie_q z$ is a strictly decreasing continuous function of z when $z_1 \in (-q, 0)$.*

Given $q \in \mathbb{R}$ and X_1, $X_2 \subset \mathbb{R}$, we put $X_1 \| X_2 = \{x_1 \| x_2 : x_1 \in X_1, x_2 \in X_2\}$ and $X_1 \bowtie_q X_2 = \{x_1 \bowtie_q x_2 : x_1 \in X_1, x_2 \in X_2\}$. We may use Lemma 2.6 to determine what happens when parallel and series reduction are applied to intervals.

LEMMA 2.7. *Let q, c_1, d_1, c_2, $d_2 \in \mathbb{R}$, and put $X_1 = (c_1, d_1)$ and $X_2 = (c_2, d_2)$.*

(a) *If c_1, $c_2 < -1$ and d_1, $d_2 > -1$, then*

$$X_1 || X_2 = (\min\{c_1 || d_2, d_1 || c_2\}, \max\{c_1 || c_2, d_1 || d_2\}).$$

(b) *If c_1, $c_2 < -q$ and $-q < d_1$, $d_2 < -q/2$, then*

$$X_1 \bowtie_q X_2 = (\min\{c_1 \bowtie_q c_2, d_1 \bowtie_q d_2\}, \max\{c_1 \bowtie_q d_2, d_1 \bowtie_q c_2\}).$$

3. Main result. Our inductive proof of Theorem 1.1 is based on using local operations, such as parallel and series reduction of edges incident to vertices on C, to transform G to a smaller nearly 3-connected plane graph. This requires us to consider weighted edges: we use the multivariate Tutte polynomial $Z_G(q, \mathbf{w})$, where $q \in (1, \gamma]$, $w_e = -1$ when e is not incident to a vertex of C, $w_e \in W_1(q)$ when e is incident to a vertex of C but $e \notin E(C)$, and $w_e \in W_2(q)$ when $e \in E(C)$, for suitably chosen intervals $W_1(q)$, $W_2(q) \subseteq (-2, 0)$. The intervals W_1 and W_2 are illustrated in Figure 3.1. Precise definitions are given below.

3.1. The interval W_1. We will require W_1 to have the properties that $W_1 \| W_1 \subseteq W_1$ and $W_1 \bowtie_q [-1] \subseteq W_1$; see Lemma 3.1 below. We accomplish this by taking $W_1 = (a_1, b_1)$, where $a_1 = (-1) \bowtie_q b_1$ and b_1 is the smallest real root of the equation $w = [(-1) \bowtie_q w] \| [(-1) \bowtie_q w]$. This gives $-w(w^2 + w(2q - 1) + q^2 - 1)(q - 1 + w)^{-2} = 0$, and hence

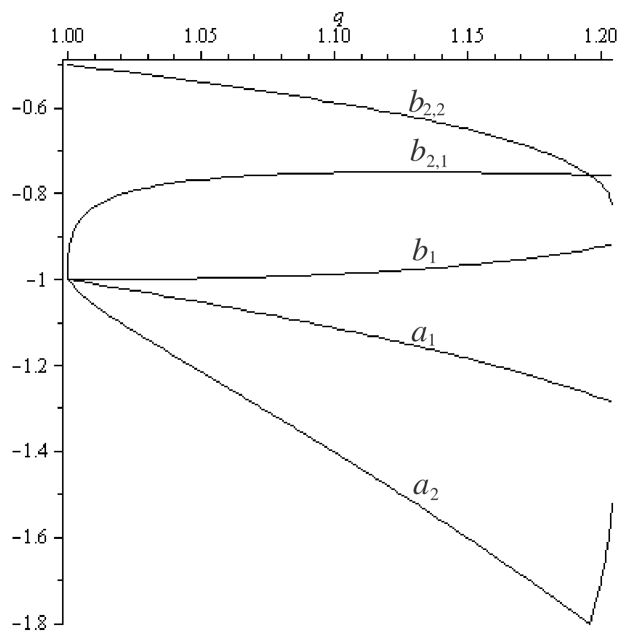

FIG. 3.1. *The intervals* $W_1 = (a_1, b_1)$ *and* $W_2 = (a_2, b_2)$ *for* $1 < q \le \gamma = 1.204\ldots,$ *where* $b_2 = \min\{b_{2,1}, b_{2,2}\}$.

$$b_1 = -q + \frac{1}{2} - \frac{1}{2}\sqrt{5-4q}$$

and

$$a_1 = \frac{-3+\sqrt{5-4q}}{2}.$$

The interval W_1 is defined for all $q \in (1, 1.25]$, although in the main result W_1 is considered for the interval $(1, \gamma]$ only.

3.2. The interval W_2. We will require W_2 to have the property that $Z_{K_3}(q, w_1, w_2, w_3) < 0$ for all $w_1,\ w_2,\ w_3 \in W_2$, and also that $W_2 \bowtie_q W_2 \subseteq W_2$ and $(W_1 \bowtie_q W_1) \| W_2 \subseteq W_2$; see Lemma 3.1 below. We accomplish this by taking $W_2 = (a_2, b_2)$, where $a_2 = b_2 \bowtie_q b_2$, $b_2 = \min\{b_{2,1}, b_{2,2}\}$, and $b_{2,1}$, $b_{2,1}$ are defined as follows. We let $b_{2,2}$ be the unique real root of the equation $Z_{K_3}(q, w, w, w) = 0$. We have $Z_{K_3}(q, w, w, w) = q(w^3 + 3w^2 + 3qw + q^2)$, and hence

$$b_{2,1} = -1 + (q-1)^{1/3} - (q-1)^{2/3} = -\frac{q}{1+(q-1)^{1/3}}.$$

We let $b_{2,2}$ be the largest real root of the equation $w = [w \bowtie_q w] \| [b_1 \bowtie_q b_1]$. We have $w - ([w \bowtie_q w] \| [b_1 \bowtie_q b_1]) = g(w)(q + 2b_1)^{-1}(q + 2w)^{-1}$, where

$$g(w) = (b_1^2 - 2b_1 - q)w^2 + (2b_1^2 - 2b_1 q - q^2)w + b_1^2 q.$$

It can be seen that $g(w) = 0$ has real roots if and only if $q \le \gamma$. It can also be seen that the two curves $b_{2,1}$ and $b_{2,2}$ have a unique point of intersection which occurs when

$g(b_{2,1}) = 0$. Solving for q, we obtain $q = \tau = 1 + \theta^3 \sim 1.1956\ldots$, where $\theta \sim 0.5805\ldots$ is a root of $x^{14} - 2x^{12} + 6x^{11} - x^{10} - 10x^9 + 15x^8 - 6x^7 - 10x^6 + 15x^5 - 12x^4 + 7x^3 - 4x^2 + 3x - 1 = 0$. We have $b_{2,1} < b_{2,2}$ for $1 < q < \tau$ and $b_{2,1} > b_{2,2}$ for $\tau < q \leq \gamma$. Thus,

$$b_2 = \begin{cases} b_{2,1} & \text{for } q \in (1, \tau), \\ b_{2,2} & \text{for } q \in [\tau, \gamma]. \end{cases}$$

The interval W_2 is defined for all $q \in (1, \gamma]$.

The intervals W_1, W_2 are constructed to be the largest subintervals of $(-\infty, 0)$ which have the properties described in the following two lemmas. The proofs of these lemmas use elementary calculus. They are given in two appendices.

LEMMA 3.1. *Let $q \in (1, 1.25]$. Then*

(a) $a_1 < -q < -1 < b_1$;

(b) $W_1 || W_1 \subseteq W_1$;

(c) $W_1 \bowtie_q [-1] \subseteq W_1$.

LEMMA 3.2. *Let $q \in (1, \gamma]$. Then*

(a) $a_2 < a_1 < -q < -1 < b_1 < b_2 < -q/2$;

(b) $W_1 \subseteq W_2$;

(c) $W_2 \bowtie_q W_2 \subseteq W_2$;

(d) $(W_1 \bowtie_q W_1) || W_2 \subseteq W_2$;

(e) $W_1 || W_2 \subseteq W_2$.

When $q > \gamma$ no interval $W_2 = (a_2, b_2)$ can satisfy the conclusions of Lemma 3.2. This is analogous to the result in [7] that, for $q > 32/27$, no nonempty interval $W \subseteq (-\infty, 0)$ can satisfy $W \| W \subseteq W$ and $W \bowtie_q W \subseteq W$.

We will use the properties of W_1 and W_2 described in Lemmas 3.1 and 3.2 to prove our promised weighted generalization of Theorem 1.1.

THEOREM 3.3. *Let $(G, \mathbf{w})$ be a weighted nearly 3-connected plane graph on n vertices, and let C be the bounding cycle of its outer face. Let $q \in (1, \gamma]$, where $\gamma = (5 - \rho^2)/4 \sim 1.2040\ldots$ and $\rho \sim 0.4289\ldots$ is a root of $x^{10} - 8x^9 + 12x^8 + 40x^7 - 82x^6 - 72x^5 + 68x^4 + 56x^3 + 313x^2 - 16x - 56 = 0$. Suppose that*

(a) *$w_e = -1$ for all $e \in E(G - C)$;*

(b) *$w_e \in W_1$ for all $e \in E(G) - E(C) - E(G - C)$;*

(c) *$w_e \in W_2$ for all $e \in E(C)$.*

Then $(-1)^n Z_G(q, \mathbf{w}) > 0$.

Proof. Suppose the theorem is false and choose a counterexample (G, w) with as few edges as possible. We can use the parallel reduction formula Lemma 2.5(a), and the facts that $W_1 \| W_1 \subseteq W_1$ and $W_1 \| W_2 \subseteq W_2$ by Lemmas 3.1(b) and 3.2(e), to deduce that no edge of G is parallel to an edge in $E(G) - E(C)$. (We do not have $W_2 || W_2 \subseteq W_2$, so we cannot deduce that $|E(C)| \neq 2$.)

Claim 3.4. $G \neq K_2^{(2)}$.

Proof. Suppose $G = K_2^{(2)}$. Then $Z_G(q, w_1, w_2) = q(q + w_1 + w_2 + w_1 w_2)$, so we need to show that $f(w_1, w_2) = q + w_1 + w_2 + w_1 w_2 > 0$ for w_1, $w_2 \in W_2$. Recall that $b_{2,1} = -1 + (q-1)^{1/3} - (q-1)^{2/3}$. Let $a_{2,1} = b_{2,1} \bowtie_q b_{2,1}$. Since $-q < b_1 \leq b_{2,1}$, Lemma 2.6(b) implies that $a_{2,1} \leq a_2$, and hence $W_2 \subseteq (a_{2,1}, b_{2,1})$. We will consider the extreme values of f for w_1, $w_2 \in [a_{2,1}, b_{2,1}]$. Since $\frac{\delta f}{\delta w_1} = 1 + w_2$, f changes monotonically with w_1 when w_2 is fixed. It follows that we need only consider what happens when $w_1 \in \{a_{2,1}, b_{2,1}\}$, and similarly $w_2 \in \{a_{2,1}, b_{2,1}\}$. We have $f(w, w) = q + (1 + w)^2 - 1 > 0$ since $q > 1$. Thus we need only consider $f(a_{2,1}, b_{2,1})$. Since $a_{2,1} = b_{2,1} \bowtie_q b_{2,1}$, we have

$$qf(a_{2,1}, b_{2,1}) = Z_G(a_{2,1}, b_{2,1}) = \frac{1}{q + 2b_{2,1}} Z_{K_3}(b_{2,1}, b_{2,1}, b_{2,1}) = 0$$

by the definition of $b_{2,1}$. Since $f(w_1, w_2)$ is strictly increasing with w_1 and strictly decreasing with w_2 when the point (w_1, w_2) is close to the point $(a_{2,1}, b_{2,1})$, we may deduce that $f(w_1, w_2) > 0$ for all w_1, w_2 in the open interval $(a_{2,1}, b_{2,1})$. ∎

Claim 3.5. Let F, H be connected subgraphs of G with $G = F \cup H$, $E(F) \cap E(H) = \varnothing$, $V(F) \cap V(H) = \{u, x\}$, and $C \subseteq H$. Suppose that $u \in V(C)$ and that F is nonseparable. Let $F + h$ be obtained by adding a new edge $h = ux$ to F, and put $F_{ux} = (F + h)/h$. Suppose that either F_{ux} is nonseparable or $x \notin V(C)$. Then the effective weight w_F of F in $(G, \mathbf{w})$ is contained in W_1.

Proof. We use an inner induction on $|E(F)|$. If $E(F) = \{e\}$, then $w_F = w_e \in W_1$ by a hypothesis of the theorem. Hence, we may suppose that $|E(F)| \geq 2$, and thus F is nonseparable. Let $L_1, L_2, \ldots, L_t$ be the ux-bridges in F. Since $W_1 || W_1 \subseteq W_1$ by Lemma 3.1(b), it will suffice to show that the effective weight w_{L_i} of L_i in $(G, \mathbf{w})$ is contained in W_1 for all $1 \leq i \leq t$. This is clearly true if $|E(L_i)| = 1$, and hence we may assume that $|E(L_i)| \geq 2$.

Suppose $t = 1$. Then $L_1 = F$. By hypothesis, F and $F + h$ are both nonseparable; the fact that $t = 1$ implies that F_{ux} is also nonseparable. Lemma 2.2(b) now implies that F, $F + h$, and F_{ux} are all nearly 3-connected. By Lemma 2.4, we have

$$w_F = \frac{(q-1)Z_F(q, \mathbf{w}|_F)}{Z_{F+h}(q, \mathbf{w}|_F, -1)} - q, \tag{3.1}$$

$$= \frac{(q-1)Z_{F_{ux}}(q, \mathbf{w}|_F)}{Z_{F+h}(q, \mathbf{w}|_{F-e}, -1)} - 1. \tag{3.2}$$

By the outer induction $Z_F(q, \mathbf{w}|_F)/Z_{F+h}(q, \mathbf{w}|_F, -1) > 0$, and hence $w_F > -q$ by (3.1). Similarly $Z_{F_{ux}}(q, \mathbf{w}|_F)/Z_{F+h}(q, \mathbf{w}|_F, -1) < 0$, and hence $w_F < -1$ by (3.2). Thus, $w_F \in W_1$.

Hence, we may assume that $t \geq 2$. This implies that F_{uv} is separable and hence, by hypothesis, $x \notin V(C)$. It also implies that L_i has fewer edges than F. Hence, if L_i is nonseparable, then we have $w_{L_i} \in W_1$ by the inner induction. Thus we may assume that L_i is separable. The hypotheses that all 2-vertex-cuts of G intersect C and that $x \notin V(C)$ now imply that L_i has exactly two blocks, B_1, B_2, with $u \in V(B_1)$, $x \in V(B_2)$, and $|V(B_2)| = 2$. Then the effective weight $w_{B_1} \in W_1$ by the inner induction, and $w_{B_2} = -1$ since $|E(B_2)| = 1$ and B_2 is not incident with a vertex of C. Thus,

$$w_{L_i} = w_{B_1} \bowtie_q [-1] \in W_1 \bowtie_q [-1] \subseteq W_1$$

by Lemma 3.1(c). ∎

Claim 3.6. Each vertex of C has degree at least 3.

Proof. Suppose C contains a vertex y of degree 2, and let $e_1 = yu$, $e_2 = yv$, be the edges incident with y. Then $e_1, e_2 \in E(C)$. Let $H = G - y$, and let $H + f$ be obtained from H by adding a new edge $f = uv$ in the outer face of H. Then the bounding cycle of the outer face of $H + f$ is $C' = C - y + f$. It is not difficult to see that $H + f$ is nonseparable, and we may now use Lemma 2.2(a) to deduce that $H + f$ is nearly 3-connected (we have $H + f = G/e_2$). By Lemma 2.5(b) we have $Z_G(q, \mathbf{w}) = (q + w_{e_1} + w_{e_2})Z_{H+f}(q, \mathbf{w}|_H, w_{e_1} \bowtie_q w_{e_2})$. Since w_{e_1}, $w_{e_2} \in W_2$, Lemma 3.2(a) implies that $q + w_{e_1} + w_{e_2} < 0$, and Lemma 3.2(c) gives $w_{e_1} \bowtie_q w_{e_2} \in W_2$. We may now use induction to deduce that $(-1)^n Z_G(q, \mathbf{w}) > 0$. $\square$

If $|V(C)| = 2$, then $G - e$ is nonseparable for each edge $e = uv \in E(C)$. On the other hand, if $|V(C)| \geq 3$, then G is simple and, by Claim 3.6 and Lemma 2.1, we may choose $e = uv \in E(C)$ such that $G - e$ is 2-connected. We may now use Lemma 2.2(a) to deduce that $G - e$ is nearly 3-connected. If G/e were nonseparable, then Lemma 2.2(a) would imply that G/e is nearly 3-connected, and we could use Lemma 2.3 and induction to deduce that $(-1)^n Z_G(q, \mathbf{w}) > 0$. Hence, G/e is separable.

Let C^* be the face of G which contains e and is distinct from C. As noted at the beginning of the proof, no edge in $E(G) - E(C)$ can be parallel to e. Hence, $|V(C^*)| \geq 3$. Let B be the C-bridge of G which contains $V(C^*) - \{u, v\}$. Since G/e is separable, the set of attachments of B on C is $\{u, v\}$. Let $F = B + e$ and $H = G - (V(F) - \{u, v\}) - E(F)$. Note that, if $|E(H)| \geq 2$, then $\{F, H\}$ is a 2-separation of G.

Since G is nonseparable, F and F/e are both nonseparable. Lemma 2.2(b) now implies that F is nearly 3-connected. Since e belongs to the outer face of F, we can apply Lemma 2.2(a) to F to deduce that F/e is also nearly 3-connected. Let w_F be the effective weight of F in $(G, \mathbf{w})$.

Claim 3.7. $w_F \in W_2$.

Proof. We first suppose that $F - e$ is nonseparable. Since F/e is nonseparable, we may apply Claim 3.5 to $F - e$ to deduce that $w_{F-e} \in W_1$. Hence,

$$w_F = w_{F-e} || w_e \in W_1 || W_2 \subseteq W_2$$

by Lemma 3.2.(e).

Thus we may assume that $F - e$ is separable. Let B_1 and B_2 be the blocks of $F - e$ which contain u and v, respectively. By Claim 3.5, we have w_{B_1}, $w_{B_2} \in W_1$. If $F - e = B_1 \cup B_2$, this gives

$$w_F = (w_{B_1} \bowtie_q w_{B_2}) || w_e \in (W_1 \bowtie_q W_1) || W_2 \subseteq W_2$$

by Lemma 3.2(d). Thus we may suppose that $F - e \neq B_1 \cup B_2$. The fact that G is nearly 3-connected now implies that $F - e$ has exactly three blocks B_1, B_2, B_3 and that $|E(B_3)| = 1$. Then $w_{B_3} = -1$, and hence

$$w_{F-e} = (w_{B_1} \bowtie_q w_{B_3}) \bowtie_q w_{B_2} \in (W_1 \bowtie_q [-1]) \bowtie_q W_1 \subseteq W_1 \bowtie_q W_1$$

by Lemma 3.1(c). Thus,

$$w_F = w_{F-e} || w_e \in (W_1 \bowtie_q W_1) || W_2 \subseteq W_2$$

by Lemma 3.2(d). $\square$

Let $(F + h, \mathbf{w}|_F, -1)$ be obtained from $(F, \mathbf{w}|_F)$ by adding a new edge $h = uv$ to F with weight -1, and let $(H + e_F, \mathbf{w}|_H, w_F)$ be obtained from $(H, \mathbf{w}|_H)$ by adding an edge $e_F = uv$ with weight w_F. By Lemma 2.4,

$$Z_G(q, w) = \frac{Z_{F+h}(q, \mathbf{w}|_F, -1)}{q(q-1)} Z_{H+e_F}(q, \mathbf{w}|_H, w_F). \tag{3.3}$$

By Lemma 2.5(a), $Z_{F+h}(q, \mathbf{w}|_F, -1) = Z_F(q, \mathbf{w}|_{F-e}, -1)$. Substituting into (3.3), we obtain

$$Z_G(q, \mathbf{w}) = \frac{Z_F(q, \mathbf{w}|_{F-e}, -1)}{q(q-1)} Z_{H+e_F}(q, \mathbf{w}|_H, w_F).$$

Since $q > 1$ and $|V(F)| + |V(H)| = n + 2$, we may apply induction to $(F, q, \mathbf{w}|_{F-e}, -1)$ and $(H + e_F, q, \mathbf{w}|_H, w_F)$ to deduce that $(-1)^n Z_G(q, \mathbf{w}) > 0$. ∎

Theorem 1.1 follows immediately from Theorem 3.3.

4. A recursive construction. We use $K_n(v_1, v_2, \ldots, v_n)$ to denote a complete graph with vertices $v_1, v_2, \ldots, v_n$. We will construct an infinite family of nearly 3-connected plane graphs with chromatic roots tending to γ. We use the *diamond operation* which replaces an edge uv of a graph G by a 4-cycle uz_1vz_2u, where $z_1, z_2 \notin V(G)$. As a first step we construct a sequence of 2-rooted plane graphs $F_1(x, y), F_2(x, y), \ldots$ with root vertices x, y such that every 2-vertex-cut of $F_i(x, y) + xy$ contains x, where $F_i(x, y) + xy$ is the graph obtained from $F_i(x, y)$ by adding a new edge joining x and y, and such that the effective weights of the $F_i(x, y)$ tend to b_1 from below. We let $F_1 = K_2(x, y)$. For $i \geq 2$, we construct F_i from F_{i-1} by performing the diamond operation on each edge incident to x; see Figure 4.1. Let w_i be the effective weight of F_i when all edge weights are equal to -1.

LEMMA 4.1. *Suppose* $q \in (1, 1.25]$. *Then*

(a) $-1 \leq w_i < b_1$ *for all* $i \geq 1$;

(b) $w_i > w_{i-1}$ *for all* $i \geq 2$;

(c) $\lim_{n\to\infty} w_n = b_1$.

Proof.

(a) We have $w_1 = -1 < b_1$. Furthermore, for any $i \geq 2$,

$$w_i = [w_{i-1} \bowtie_q (-1)] \| [w_{i-1} \bowtie_q (-1)]. \tag{4.1}$$

Since, for any $w \in \mathbb{R}$, we have $w \| w = (w+1)^2 - 1 \geq -1$, this immediately implies that $w_i \geq -1$. Furthermore, since $w_1 \in W_1$, we may use (4.1), Lemma 3.1 (b),(c), and induction to deduce that $w_i \in (W_1 \bowtie_q [-1]) \| (W_1 \bowtie_q [-1]) \subset W_1$. Hence, $w_i < b_1$.

(b) Let $f(w) = [w \bowtie_q (-1)] \| [w \bowtie_q (-1)]$. Then

$$f(w) - w = -w(w^2 + w(2q-1) + q^2 - 1)(q - 1 + w)^{-2}.$$

Since b_1 is the smallest real root of $w^2 + w(2q-1) + q^2 - 1 = 0$, we have $f(w) - w > 0$ for all $w < b_1$. We can now use (a) to deduce that $w_i - w_{i-1} = f(w_{i-1}) - w_{i-1} > 0$.

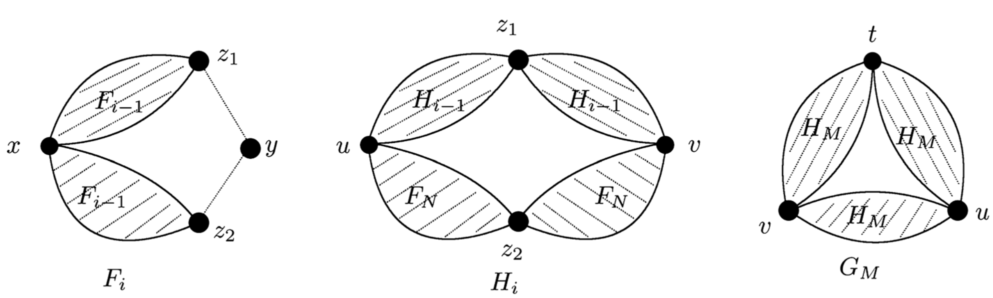

FIG. 4.1. *The graphs F_i, H_i for $i \geq 2$ and the graph G_M.*

(c) We have shown that $\{w_i\}$ is a strictly increasing sequence which is bounded above by b_1. This implies that $\{w_i\}$ must converge to a limit α which evidently must satisfy $\alpha \leq b_1$ and $f(\alpha) = \alpha$. Hence, $\alpha = b_1$. ∎

Let $\mathcal{F}$ denote the family of nearly 3-connected plane graphs. For $G \in \mathcal{F}$ we use $Z_G(q, w, -1)$ to denote the multivariate Tutte polynomial for G with all edge weights on the outer face equal to w and all other edge weights equal to -1.

Let $F(x, y)$ be a 2-rooted nonseparable plane graph with root vertices x, y lying on the outer face of F, and with the property that every 2-vertex-cut of $F(x, y) + xy$ contains x. Let w_F denote the effective weight of F when all edges have weight -1. We use F to define a modified diamond operation on $\mathcal{F}$. Given $G \in \mathcal{F}$ with outer face C, the *F-diamond operation* replaces an edge uv on C by a graph $D = D_1 \cup D_2 \cup D_3 \cup D_4$, where $V(D) \cap V(G) = \{u, v\}$, $D_1 = K_2(u, z_1)$, $D_2 = K_2(v, z_1)$, $D_3 = F(v, z_2)$, $D_4 = F(u, z_2)$, and z_1 lies on the outer face of the new graph. Let $\lozenge_F(G)$ denote the graph obtained by applying the F-diamond operation to every edge of C. The fact that every 2-vertex-cut of $F_i(x, y) + xy$ contains x implies that $\lozenge_F(G) \in \mathcal{F}$. We may use Lemma 2.4 to express $Z_{\lozenge_F(G)}(q, w, -1)$ in terms of $Z_G(q, \diamond(w), -1)$, where $\diamond(w) = (w \bowtie_q w) \| (w_F \bowtie_q w_F)$. We first replace both copies of F in each copy of D in $\lozenge_F(G)$ by edges of weight w_F, and then we replace each resulting C_4 by an edge of weight $\diamond(w)$. Lemma 2.4 then gives

$$Z_{\lozenge_F(G)}(q, w, -1) = (q + 2w)^m (q + 2w_F)^m \frac{(Z_{F+h}(q, -1))^{2m}}{(q(q-1))^{2m}} Z_G(q, \diamond(w), -1), \tag{4.2}$$

where $F + h$ is obtained from F by adding a new edge $h = xy$, and $m = |E(C)|$.

THEOREM 4.2. *For each $q \in (\gamma, 1.25]$ there exists a nearly 3-connected loopless plane graph G such that $P_G(q_1) = 0$ for some $q_1 \in (\gamma, q]$.*

Proof. Choose $q \in (\gamma, 1.25]$, and let

$$h(w, b) = (b^2 - 2b - q)w^2 + (2b^2 - 2bq - q^2)w + b^2 q.$$

We have $2b_1 + q = -q + 1 - \sqrt{5 - 4q} < 0$ for all $q \in (1, 1.25)$. The definition of W_1 and the fact that $q > \gamma$ now imply that $h(w, b_1) = g(w) > 0$ for all $w \in \mathbb{R}$. Thus, $(2b_1^2 - 2b_1 q - q^2)^2 - 4b_1^2(b_1^2 - 2b_1 - q) < 0$. By continuity, there exists $b_0 \in (-1, b_1)$ such that $h(w, b) > 0$ for all $w \in \mathbb{R}$ and all $b \in (b_0, b_1)$. Since $\{w_i\}$ is a strictly increasing sequence which converges to b_1 by Lemma 4.1, we may choose a positive integer $N = N(q)$ such that $w_N \in (b_0, b_1)$.

Recall that w_N is the effective weight of the 2-rooted graph $F_N(x, y)$. We use the F_N-diamond operation to give a recursive construction for our required graph G. Put $G_1 = K_3(t, u, v)$ and, for $i \geq 2$, let $G_i = \lozenge_{F_N}(G_{i-1})$. We will show that $P_{G_i}(q_1) = 0$ for some $i \geq 1$ and some $q_1 \in (\gamma, q]$.

Let $H_1(u, v) = K_2(u, v)$ and, for $i \geq 2$, let $H_i(u, v)$ be the 2-rooted subgraph of G_i obtained from $H_{i-1}(u, v)$ by applying the F_N-diamond operation to all edges of $H_{i-1}(u, v)$ which lie on the outer face of G_{i-1}. For $i \geq 2$ we have $H_i = A \cup B \cup C \cup D$, where $A = H_{i-1}(u, z_1)$, $B = H_{i-1}(v, z_1)$, $C = F_N(v, z_2)$, and $D = F_N(u, z_2)$, and z_1 lies on the outer face of G_i; see Figure 4.1. Let r_i be the effective weight of the 2-rooted subgraph H_i when all edges have weight -1. We will show that $r_M \geq -q/2$ for some $M \geq 1$.

Suppose to the contrary that $r_i < -q/2$ for all $i \geq 1$. We have $r_1 = -1$ and, since the effective weight of $F_N(x, y)$ is w_N, $r_{i+1} = [r_i \bowtie_q r_i] \| [w_N \bowtie_q w_N]$ for all $i \geq 1$. Hence,

$$r_{i+1} - r_i = [r_i \bowtie_q r_i] \| [w_N \bowtie_q w_N] - r_i = h(r_i, w_N)(q + 2r_i)^{-1}(q + 2w_N)^{-1} > 0$$

since $w_N \in (b_0, b_1)$. Thus $\{r_i\}$ is a strictly increasing sequence which is bounded above by $-q/2$. This implies that $\{r_i\}$ must converge to a limit β which evidently must satisfy $h(\beta, w_N) = 0$. This is impossible since $h(w, w_N) > 0$ for all $w \in \mathbb{R}$. Hence, we may choose a positive integer M such that $r_M \geq -q/2$.

Consider the graph G_M. We have $G_M = R \cup S \cup T$, where $V(R) \cap V(S) = \{t\}$, $V(S) \cap V(T) = \{u\}$, $V(T) \cap V(R) = \{v\}$, and each of R, S, T is a 2-rooted subgraph isomorphic to $H_M(x, y)$; see Figure 4.1. We may use (4.2) and the fact that G_i has an even number of edges on its outer face for all $i \geq 2$ to deduce that

$$P_{G_M}(q) = Z_{G_M}(q, -1) = (\text{nonnegative prefactor}) Z_{K_3}(q, r_M).$$

We have $Z_{K_3}(q, r_M) = q(r_M^3 + 3r_M^2 + 3qr_M + q^2) > 0$ since $r_M \geq -q/2$ and $q \in (1, 2)$. Thus, $P_{G_M}(q) \geq 0$. On the other hand, $P_{G_M}(\gamma) < 0$ by Theorem 1.1. Continuity now implies that $P_{G_M}(q_1) = 0$ for some $q_1 \in (\gamma, q]$. $\square$

5. Open problems. It seems difficult to adapt the inductive proof technique used in this paper for the family of 3-connected graphs. There are, however, other families of "nearly 3-connected graphs" for which a similar approach may work.

CONJECTURE 5.1. *Let G be a nonseparable plane graph on n vertices, and let C be the bounding cycle of the outer face of G. Suppose that, for all 2-separations $\{G_1, G_2\}$ of G, we have $V(G_1) \cap V(G_2) \subseteq V(C)$. Then $(-1)^n P_G(q) > 0$ for all $q \in (1, \delta]$, where $\delta \sim 1.225\ldots$ is a root of $x^4 - 4x^3 + 4x^2 - 4x + 4 = 0$.*

CONJECTURE 5.2. *Let G be a nonseparable plane graph on n vertices, and let C be the bounding cycle of the outer face of G. Suppose that, for all 2-separations $\{G_1, G_2\}$ of G, we have $E(C) \cap E(G_1) \neq \varnothing \neq E(C) \cap E(G_2)$. Then $(-1)^n P_G(q) > 0$ for all $q \in (1, \rho]$, where $\rho \sim 1.430\ldots$ is a root of $3x^{10} - 19x^9 + 54x^8 - 114x^7 + 303x^6 - 831x^5 + 1704x^4 - 2540x^3 + 2400x^2 - 1344x + 512 = 0$.*

CONJECTURE 5.3. *Let G be a nonseparable graph on n vertices and x be a vertex of G. Suppose that, for all 2-separations $\{G_1, G_2\}$ of G, we have $x \in V(G_1) \cap V(G_2)$. Then $(-1)^n P_G(q) > 0$ for all $q \in (1, 1.25]$.*

The zero-free intervals given in Conjectures 5.1, 5.2, and 5.3 would be best possible. The family of graphs obtained recursively from K_3 by applying the diamond operation to each edge on the outer face satisfies the hypotheses of Conjecture 5.1 and has a chromatic root converging to δ from above. The family of graphs obtained recursively from K_3 by replacing each edge uv on the outer face by the 2-rooted subgraph F, where $V(F) = \{u, v, x_1, x_2, x_3, x_4\}$ and $E(F) = \{ux_1, x_1x_2, x_2x_3, x_3v, x_4u, x_4x_2, x_4v\}$, satisfies the hypotheses of Conjecture 5.2 and has a chromatic root converging to ρ from above. The family of graphs obtained recursively from $K_3(x, y, z)$ by applying the diamond operation to each edge incident to x satisfies the hypotheses of Conjecture 5.3 and has a chromatic root converging to 1.25 from above.

Appendix A. Proof of Lemma 3.1.

(a) This follows directly from the definitions of a_1 and b_1 at the beginning of section 3.1.

(b) By (a), we have $a_1 < -1 < b_1$. Lemma 2.7(a) implies that $W_1 \| W_1 = (a_1 \| b_1, \max\{a_1 \| a_1, b_1 \| b_1\})$. We have $(a_1 \| b_1) - a_1 = (a_1 + 1)b_1 > 0$, so $a_1 \| b_1 > a_1$. Similarly $(b_1 \| b_1) - b_1 = b_1(b_1 + 1) < 0$, so $b_1 \| b_1 < b_1$. We also have

$$a_1\|a_1 = ([-1] \bowtie_q b_1)\|([-1] \bowtie_q b_1) = b_1$$

by the definition of W_1. Thus, $W_1\|W_1 = (a_1\|b_1, b_1) \subseteq W_1$.

(c) Since $-1 \in (-q, 0)$, Lemma 2.7(b) implies that

$$W_1 \bowtie_q [-1] = (b_1 \bowtie_q [-1], a_1 \bowtie_q [-1]) = (a_1, a_1 \bowtie_q [-1]).$$

Furthermore,

$$a_1 \bowtie_q [-1] - b_1 = \frac{2(q-2)(q-2+\sqrt{5-4q})}{(2q-5+\sqrt{5-4q})} < 0$$

for $1 < q \leq 1.25$. Thus, $a_1 \bowtie_q [-1] < b_1$, so $W_1 \bowtie_q [-1] \subseteq W_1$. □

Appendix B. Proof of Lemma 3.2.

(a) By Lemma 3.1(a), it suffices to show that $a_2 < a_1$ and $b_1 < b_2 < -q/2$. We first show that

$$b_1 < b_2 < -q/2. \tag{B.1}$$

We consider the cases when $q \leq \tau$ and $q \geq \tau$ separately.

Case 1. $1 < q \leq \tau$. We have $b_2 + \frac{q}{2} = -\frac{q}{1+(q-1)^{1/3}} + \frac{q}{2} < 0$ for all $1 < q < 2$, so $b_2 < -\frac{q}{2}$. Furthermore,

$$\begin{aligned} b_2 - b_1 &= \frac{-q}{1+(q-1)^{1/3}} + q - \frac{1}{2} + \frac{1}{2}\sqrt{5-4q} \\ &= \frac{(q-1)^{1/3}(q\sqrt{5-4q} + 2 - q - 2(q-1)^{1/3})}{(1+(q-1)^{1/3})(1+\sqrt{5-4q})}. \end{aligned}$$

Since $2 - q - 2(q-1)^{1/3} > 0$ for $1 \leq q \leq 1.23$, we have $b_2 > b_1$.

Case 2. $\tau < q \leq \gamma$. We have $b_2 = b_{2,2}$, where $b_{2,2}$ is the largest root of $g(x) = 0$, and

$$g(x) = (b_1^2 - 2b_1 - q)x^2 + (2b_1^2 - 2b_1 q - q^2)x + b_1^2 q.$$

Let $b_{2,2}^-$ be the other root. Then $b_{2,2}^- \leq b_{2,2}$ and

$$\frac{b_{2,2}^- + b_{2,2}}{2} = -\frac{2b_1^2 - 2b_1 q - q^2}{2(b_1^2 - 2b_1 - q)}.$$

We have $2b_1 + q = -q + 1 - \sqrt{5-4q} < 0$ for $1 < q < 1.25$. Thus,

$$b_1^2 - 2b_1 - q > 0. \tag{B.2}$$

Hence, the coefficient of x^2 in $g(x)$ is positive, and we have $g(x) \leq 0$ for $b_{2,2}^- \leq x \leq b_{2,2}$. Thus, to show that $b_{2,2} < -q/2$, it will suffice to show that $(b_{2,2}^- + b_{2,2})/2 < -q/2$ and $g(-q/2) > 0$. We have

$$\frac{b_{2,2}^- + b_{2,2}}{2} + \frac{q}{2} = -\frac{2b_1^2 - 2b_1 q - q^2}{2(b_1^2 - 2b_1 - q)} + \frac{q}{2} = \frac{b_1^2(2-q)}{2(q - b_1^2 + 2b_1)} < 0$$

by (B.2), and

$$g(-q/2) = q^2(q + 2b_1 + b_1^2)/4 = q^2(q - 1 + (b_1 + 1)^2)/4 > 0.$$

Hence, $b_{2,2} < -q/2$.

It remains to show that $b_{2,2} > b_1$. We have

$$\text{(B.3)} \qquad \frac{b_{2,2}^- + b_{2,2}}{2} - b_1 = \frac{q^2 - 6b_1^2 + 2b_1^3}{2(q - b_1^2 + 2b_1)}.$$

Furthermore,

$$q^2 - 6b_1^2 + 2b_1^3 = -(5 - 4q^2 + 2q^3) - (3q^2 - 1 + 2q)\sqrt{5 - 4q}.$$

Since $5 - 4q^2 + 2q^3 > 0$ and $3q^2 - 1 + 2q > 0$ for $q \in (1, 1.25)$, we have $q^2 - 6b_1^2 + 2b_1^3 < 0$. We may now use (B.2) and (B.3) to deduce that $(b_{2,2}^- + b_{2,2})/2 - b_1 > 0$, and hence $b_{2,2} \geq (b_{2,2}^- + b_{2,2})/2 > b_1$.

This completes the proof of (B.1). Using (B.1), Lemma 2.6(b), and the definitions of a_1, a_2, we have

$$a_2 = b_2 \bowtie_q b_2 < (-1) \bowtie_q b_1 = a_1.$$

(b) This follows immediately from (a).

(c) Since $a_2 < -q < b_2$ by (a), Lemma 2.7(b) implies that

$$W_2 \bowtie_q W_2 = (\min\{a_2 \bowtie_q a_2, b_2 \bowtie_q b_2\}, a_2 \bowtie_q b_2).$$

We have $b_2 \bowtie_q b_2 = a_2$ by the definition of a_2. Furthermore, (a) implies that $(a_2 \bowtie_q a_2) - a_2 = -a_2(a_2 + q)/(q + 2a_2) > 0$ and $(a_2 \bowtie_q b_2) - b_2 = -b_2(b_2 + q)/(q + a_2 + b_2) < 0$. Thus, $W_2 \bowtie_q W_2 = (a_2, a_2 \bowtie_q b_2) \subseteq W_2$.

(d) Since $a_1 < -q < b_1$, Lemma 2.7(b) implies that

$$W_1 \bowtie_q W_1 = (\min\{a_1 \bowtie_q a_1, b_1 \bowtie_q b_1\}, a_1 \bowtie_q b_1).$$

We have

$$(b_1 \bowtie_q b_1) - (a_1 \bowtie_q a_1) = \frac{(2 - q)\left(\sqrt{5 - 4q} - 1\right)^2\left(\sqrt{5 - 4q} - 3\right)^2}{16\left(q - 1 + \sqrt{5 - 4q}\right)\left(q - 3 + \sqrt{5 - 4q}\right)} < 0$$

for $1 < q \leq 1.25$. Hence, $b_1 \bowtie_q b_1 < a_1 \bowtie_q a_1$. We also have $(a_1 \bowtie_q b_1) - b_1 = -b_1(q + b_1)(q + a_1 + b_1)^{-1} < 0$ by (a). Thus, $a_1 \bowtie_q b_1 < b_1$. Hence,

$$\text{(B.4)} \qquad W_1 \bowtie_q W_1 \subseteq (b_1 \bowtie_q b_1, b_1).$$

Since $b_1 \in (-q, 0)$, Lemma 2.6(b) implies that $z \bowtie_q b_1$ is a strictly decreasing function of z. Hence, $b_1 \bowtie_q b_1 < (-q) \bowtie_q b_1 = -q < -1$. Since we also have $a_1 < -1 < b_1 < b_2$, we can use Lemma 2.7(a) and (B.4) to deduce that

$$(W_1 \bowtie_q W_1) \| W_2 \subseteq (\min\{(b_1 \bowtie_q b_1) \| b_2, b_1 \| a_2\}, \max\{(b_1 \bowtie_q b_1) \| a_2, b_1 \| b_2\}).$$

We have $(b_1 \| a_2) - a_2 = b_1(a_2 + 1) > 0$, and hence

(B.5) $$b_1 \| a_2 > a_2.$$

Similarly $(b_1\|b_2) - b_2 = b_1(b_2+1) < 0$, so

(B.6) $$b_1 \| b_2 < b_2.$$

We also have

$$\begin{aligned}
&[(b_1 \bowtie_q b_1)\|b_2] - a_2 \\
&= \frac{b_1^2 b_2}{q+2b_1} + \frac{b_1^2}{q+2b_1} + b_2 - \frac{b_2^2}{q+2b_2} \\
&= \frac{b_2(q+2b_1)([q-1]+[b_1+1][b_2+1]) + b_1 q(1+b_2)(b_1-b_2)}{(q+2b_1)(q+2b_2)} \\
&> 0,
\end{aligned}$$

as both the numerator and denominator are positive. Thus $(b_1 \bowtie_q b_2)\|b_2 > a_2$. It remains to show that $(b_1 \bowtie_q b_1)\|a_2 \le b_2$. From the definition of W_2, we have $a_2 = b_2 \bowtie_q b_2$ and

$$b_2 - [(b_1 \bowtie_q b_1)\|(b_2 \bowtie_q b_2)] = g(b_2)(q+2b_1)^{-1}(q+2b_2)^{-1}.$$

When $q \in [\tau,\gamma]$ we have $b_2 = b_{2,2}$, $g(b_{2,2}) = 0$, and hence $(b_1 \bowtie_q b_1)\|a_2 = b_2$. When $q \in (1,\tau)$ we have $b_2 = b_{2,1}$. Since τ is the unique root of $g(b_{2,1}) = 0$, $g(b_{2,1})$ has constant sign for $q \in (1,\tau)$. It is not difficult to check that this sign is positive. Thus, $b_2 - [(b_1 \bowtie_q b_1)\|a_2 = g(b_2)(q+2b_1)^{-1}(q+2b_2)^{-1} > 0$. So in both cases we have

(B.7) $$(b_1 \bowtie_q b_1)\|a_2 \le b_2.$$

(e) Since $a_2 < a_1 < -q < -1 < b_1 < b_2$ by (a), Lemma 2.7(a) implies that

$$W_1\|W_2 = (\min\{a_1\|b_2, b_1\|a_2\}, \max\{a_1\|a_2, b_1\|b_2\}).$$

We have $b_1\|a_2 > a_2$ and $b_1\|b_2 < b_2$ by (B.5) and (B.6), respectively. We also have $(a_1\|b_2) - a_1 = b_2(a_1+1) > 0$. Hence, $a_1\|b_2 > a_1 > a_2$. It remains to show that $a_1\|a_2 \le b_2$.

Since $b_1 \in (-q,0)$, Lemma 2.6(b) implies that $z \bowtie_q b_1$ is a strictly decreasing function of z, and hence $a_1 = (-1) \bowtie_q b_1 > b_1 \bowtie_q b_1$. Since $a_2 < -1$, Lemma 2.6 (a)

implies that $z\|a_2$ is also a strictly decreasing function of z, and hence

$$a_1\|a_2 < (b_1 \bowtie_q b_1)\|a_2 \le b_2$$

by (B.7). ☐

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