Differences in the processes of solving mathematical problems between successful and unsuccessful solvers

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Introduction

In a classroom situation when a teacher gives students mathematical problems to solve, it is often clear that some individuals will solve them faster and more efficiently than others. Learning by solving problems is a significant development of skills in mathematics. With the current emphasis on developing students' ability in mathematical problem solving, teachers in Singapore schools are beginning to see problem solving as more important than acquiring or remembering symbols, facts and procedures. Therefore if teachers can find out the mental processes of:

1. How successful problem solvers translate a given situation in understanding a problem?

2. How successful problem solvers select a suitable strategy and plan a course of action?

3. How their metacognition, *i.e. the ability to control one's own thinking,* influence success in problem solving?

4. How task irrelevant behaviours affect the unsuccessful problem solver?

then teachers can tailor their mathematics lessons and individualise instruction to help students improve their mathematical thinking and problem solving performance.

Thus to find how problem solvers process information while solving mathematical problems, an investigation was carried out on two small groups of students, successful problem solvers and unsuccessful problem solvers. They were selected from a class of 35 pre-service teachers based on their performance on a problem solving test of five non-routine mathematical problems. The first group were the top ten who completed all the problems successfully while the latter group were the bottom ten who only completed partially no more than two problems.

The "Think-Aloud" Method

So how does one make the mental processes of the subjects who are solving problems, observable? In order for the subjects to report on their mental states, they were trained to say out loud everything that comes into their mind while working on a problem. This "think aloud" method allows an investigator to collect verbal data through tape recording which are then transcribed and analysed. More details on this technique and protocol analysis can found in Foong (1993). After being trained to "think aloud" the subjects were asked to attempt two problems. No time limit was imposed and it was stressed that the main interest was in hearing them solve each problem while thinking aloud and that finding a correct solution was less important than how they arrived at their conclusions.

The Problems

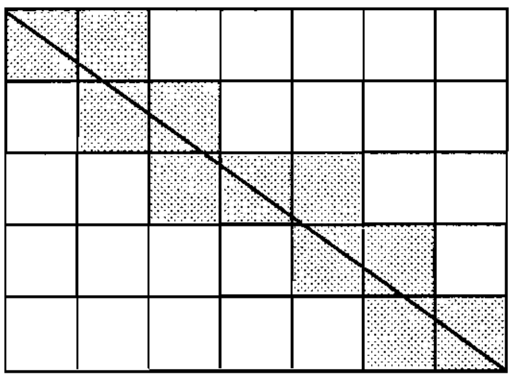
A contentious issue in the study of mathematical problem solving processes is what type of tasks should be chosen as problems. The task because it provides the stimulus for an individual to respond in a problem situation, is an essential and central component in the problem solving process. By the definition of a "problem" the task has to be "reasonably complex" but approachable. Such problems are usually not solved by simple recall even though a student may possess adequate mathematical knowledge. Two non-routine process problems which require problem solving strategies, were chosen for this investigation:

PROBLEM 1: ***CHICKEN COOP***

The Smith family wants to build a chicken coop and they have bought enough wire for 19 metres of fence and one gate that is one metre wide. They have decided to make the coop rectangular or square. What width and length would they make the coop so that the chickens have the largest possible area inside the coop?

There are two common approaches used by students to solve this problem. They either used a routine calculus procedure or a "guess and check" method based on the elementary mensuration of a rectangle. Before the appropriate strategy can be applied the problem-solvers have to picture how a chicken coop should be represented by the "fence" and the "gate" with respect to the given measurements. There are two conditions to consider for the unknowns: the constraint of a fixed perimeter and the condition that there should be a maximum area.

PROBLEM 2: ***DIAGONALS***



A diagonal of this 5 x 7 rectangle passes through 11 squares (shaded). Can you find a way of forecasting the number of squares passed through if you know the dimensions of the rectangle? How many squares will the diagonal of a 1000 x 800 rectangle pass through?

Unlike the CHICKEN COOP problem, this DIAGONALS problem is not-domain-specific as there is no algorithmic rule or procedure that one can apply directly for a solution. The problem involves situations in which, starting from the consideration of simple cases to generate data from which a pattern can be found and then formulated into a general rule. These are the important problem solving behaviours that this problem aims to elicit from the subjects.

Analysis of the Problem Solving Processes

It is generally agreed that there are four essential stages in the process of problem solving. Polya (1957) examined his own thoughts to find useful patterns of problem solving behaviours. The result was a general prescription of a four-phase model of the process: understanding the problem, devising a plan, carrying out the plan and looking back. The details of each stage included a range of problem solving behaviours. A review of the problem solving literature provided the investigator with a list of frequently recurring behaviours that could be classified into five major categories:

1. *Understanding the problem:* strategies through which a problem-solver attempts to analyse and understand the problem situation.

2. *Planning and carrying out the solution:* rule-of-thumb strategies through which a problem-solver moves towards a solution.

3. *Domain-Specific Knowledge:* the inventory of mathematical facts, procedures and skills that a problem solver is able to use in the solution process.

4. *Metacognition:* a problem-solver's awareness and monitoring progress of his or her own thinking during the task.

5. *Affective Behaviours:* self-related expressions and emotional responses that are aroused in the problem solver.

The above classification of behaviours is then used as a dictionary for analysing the differences in the mathematical problem-solving processes of the successful and unsuccessful problem solvers in this investigation.

Major Findings

1. Processes Involved in Understanding the Problem

Successful Problem Solvers translated the problem statement more correctly and more exactly than did Unsuccessful Problem Solvers.

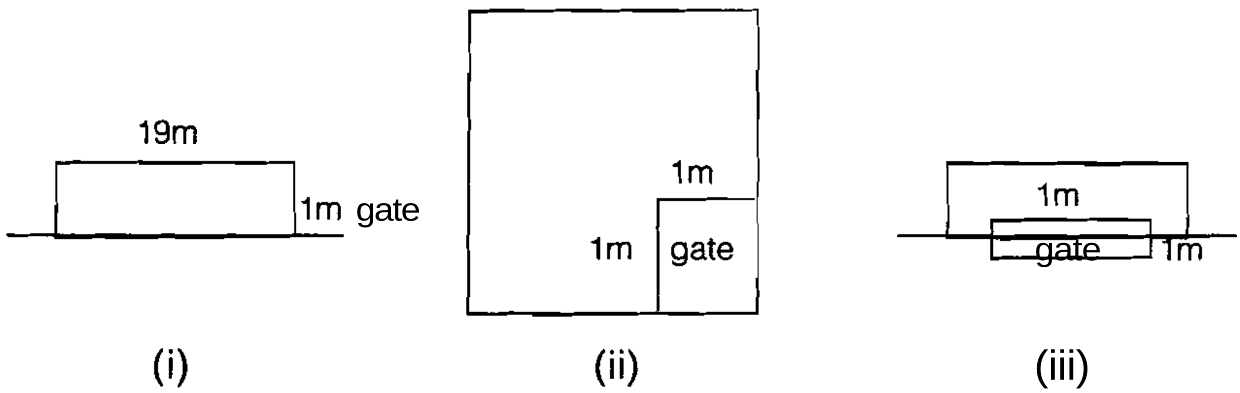
The successful solvers were not confused by superficial features but able to concentrate on the structural components of the problem. They made efficient use of diagrams to visualise and break down information. In the chicken-coop problem most of them were able to identify the two constraints in the problem i.e., the constant perimeter and the maximum area. In the diagonal problem they had no difficulty in translating the situation as finding two sub-problems, first to generalise a pattern and then provide answer for a specific dimension.

- *Unsuccessful Problem Solvers tended to attend to obvious details, translating statement by statement without having a global representation of the problem situation.*

Most of them had difficulty in getting started, reading the problem many times. They could not identify "hidden" relationships in the problem premise and easily confused by multiple elements even with the help of diagrams.

For example, eight out of ten of the non-successful solvers had wrong preconceived ideas about a chicken coop and were not able to identify the perimeter of the whole coop as 20 metres, consisting of 19 metres of fence and an additional measure of 1 metre for the gate. They concentrated on superficial features like the gate and spent a lot of time figuring out its position with the help of diagrams. They had a preconceived idea that if there was a "length" given, then there must be a "breadth" somewhere. Their diagrams shown in Figure 1, confirmed their misconceptions. As a result they disregarded the actual situation and had a vague notion of what they had to find and what the constraints were. In the diagonal problem all of them misinterpreted the problem. Six of them gave up after a few fruitless attempts to identify what it was that they had to find.

Figure 1.
Some common misrepresentations of the chicken-coop.



2. *Processes involved in Executing a Solution*

- *Successful Problem Solvers planned their solutions in more detail before carrying them out than unsuccessful solvers, who tended to be impulsive in executing a solution without complete understanding of the problem.*

The successful solvers displayed a good background knowledge of mathematics which enabled them to internalise the problem with a schema to execute a solution. Many of them recognised the chicken-coop problem as a calculus example of the 'max-min' type and applied the appropriate procedures for a solution. In the diagonal problem, after they had understood the nature of the problem which required looking for patterns, the successful solvers adopted a systematic approach by exploring a series of simple cases which they tabulated and from there formulated a guess and check strategy to generalise a rule. They showed persistence in pursuing the solution accepting wrong guesses or unexpected results as opportunities to explore further.

- *Unsuccessful Problem Solvers tended towards impulsive solutions and when in difficulty they often returned to the same incorrect method, sometimes repeatedly.*

  In the chicken-coop problem, the non-successful solvers either ignored or were unable to coordinate the two conditions in the problem. There were attempts to use a "guess and check" strategy to find the largest area but not completely. They would test two cases for instance, a square and any rectangle then deduced from there that a square was the largest. They either ignored the condition of the fixed perimeter to find area or they would find the length and width that would fit the "fence and one gate" and not the hidden unknown in the area. There was a tendency to conclude without looking back to ensure conditions were met.

  In the diagonal problem the non-successful solvers had tendencies towards an impulsive application of irrelevant mathematical algorithms or engaged in "number crunching" the given dimensions in the problem. Some actually drew a 1000 by 800 rectangle on squared paper and proceeded to count the unknown number of squares, which used up much of their time. Most of them first thought of using Pythagoras' Theorem which led them nowhere, then they often tried using ratio and proportion, then areas of triangles and even the formula "hypotenuse = opposite over adjacent".

  It showed that many of them had rote learned geometrical rules about "rectangle and diagonal" which they tried to recall to suit the circumstances as perceived by them. One subject said, "There must be a formula of some kind . . . . . I forgot how to find the hypotenuse, so I cannot use that . . so what am I suppose to do . . . . go back to ratio . . ." One subject was "stuck" by the purely visual aspect when she examined and counted the patterns of shaded squares on her diagrams from which she deduced that "there is always one square extra in the middle (row)" as a rule.

Because the thinking of the non-successful solvers was so convergent towards a single rule or formula for the solution, they had no idea about exploring to generate new cases of rectangles other than those given in the problem statement. It could also be possible that, being so used to routine problems with a single direct solution, they did not know the kind of strategies that were expected of them when such a non-routine problem like DIAGONALS was presented.

Generally most of the unsuccessful solvers tended to be "stuck" with a certain line of attack, doing mechanical computation without assessing its relevance to the task at hand. They often lacked deductive reasoning skills and jumped to hasty conclusion after trialling a few random cases. The tendency was to give up when an impasse was reached.

3. Metacognitive Processes

- *Successful Problem Solvers used more metacognitive processes which were task directed, showing greater awareness of how a things were in the solution path and where they should be going in the process.*

The problem-solving processes of successful problem solvers were monitored by regular review and reflection. In the diagonal problem which was exploratory in nature, most of the solution processes were long and it was possible that the solvers could be easily diverted if they did not constantly assess or review the situation. They used metacognitive behaviours to monitor progress, not only globally but also locally while implementing strategies. The metacognitive behaviours like "recognise new information", "self-questioning" and "recognise errors" were used by successful problem solvers in spotting patterns. They were simultaneously evaluating new alternatives to conjectures or the correctness of procedures. Examples of such verbalisations were:

> "Oh dear, not very accurate, have to be accurate otherwise it affects my answer."
>
> "Something is wrong with my diagonal . . . "
>
> "Actually I don't have to consider 6 x 12 . . . just take small numbers and see if I can generalise from a few specific cases . . . OK, let's draw a table and see . . ."

In general the successful problem solvers showed that they were reflective problem solvers who were consciously assessing the current state of their solution paths and were able to discover valuable information while trying several strategies to reach a solution.

On the other hand, the non-successful solvers also displayed some metacognitive behaviours but to a much lesser extent. The substance of their monitoring behaviours was primarily an awareness of their own confusion and uncertainty. Examples of common verbalizations of such behaviours among the non-successful solvers were:

> "I found that this problem is quite difficult, if I draw 1000 by 800 squares it's going to be tough.",
>
> "Do I work from here? . . . It cannot be that simple, you know."
>
> "Cannot use that, it's not logical . . .then what do I do?" and so on.

Much of the self-questioning was rhetoric in nature. Many of them spent an average of 11.4 minutes on the problem but their solution paths were undirected and rambling.

4. Affective Behaviours

- *Negative emotional expressions such as frustration and confusion were found to be more frequent in the unsuccessful problem solvers.*

  Some of the first reactions to a problem after it was read by the insuccessful solvers were:

  > "Oh dear, this looks so difficult."
  >
  > "Very, very difficult."
  >
  > "Oh! My God, what's this." etc . .

  These behaviours showed awareness of task facility, they were indicative of an emotional response such as anxiety that is aroused in a subject when an insurmountable task was perceived. With such initial negative perceptions of the problem, these subjects then proceeded against a background of difficulties that often aroused negative self-evaluative behaviours. Some of their expressions of frustrations were:

  > "I think I have enough of this!"
  >
  > "This is a headache, Oh, God!"
  >
  > "Aiyah . . . really don't know what is this, sounds interesting but just don't know where to start!"
  >
  > "No, I'm not thinking . . . oh, dear" and finally
  >
  > "Ah . . . can't do, give up."

Generally, during the solution process when they failed to obtain some results, they became easily irritated and their behaviours became self-directed and task-irrelevant. They expressed a lack of self confidence and would then either continue in a rambling manner or they gave up thus avoiding further frustration.

Conclusion

The findings in this study have shown that although the adult subjects, who were pre-service teachers, had the prerequisite mathematics education qualifications, many of them among the unsuccessful solvers, were not able to apply their mathematical knowledge to solving problems and they displayed the characteristic behaviours of "naive" problem solvers. The results indicate that there should be strong support for both the curriculum and teachers to provide students with rich situational problern solving opportunities. Problem solving activities should be viewed as learning situations for introducing and developing instruction for learning mathematical concepts and skills and towards producing students who will be better equipped to use their mathematical knowledge to solve all mathematical problems including the non-routine.

The findings of this study have also shown that the unsuccessful problem solvers lacked knowledge of a set of useful strategies to help them approach an unfamiliar problem and that the ability to also reflect on the task at hand was often missing. If problem solvers could get into the habit of self-questioning such as "How can I understand this problem?" and discussing with peers why certain approaches to a problem were chosen, they might begin to see the task of problem solving somewhat differently. The metacognitive knowledge of when and which strategy to apply and of how to tell whether they are working is absolutely essential for improvement in problem solving performance.

In a later study, these same unsuccessful solvers were given training in a metacognitive approach to using heuristic strategies in solving non-routine mathematical problems. Pair and small group techniques that differed from the traditional classroom style of teacher exposition were used. Here the subjects were placed in a non-threating environment that emphasised certain specific strategies and were given a concentrated course of relevant practice problems.

Despite the limited period of training, the majority of the subjects did show consistent improvement through a series of "thinking aloud" interviews in their use of the heuristic and evaluative strategies. An implication arising from this "teaching experiment" is that problem solving processes involving heuristics and metacognition can be taught explicitly. Subjects can learn to use these strategies more effectively together with the mathematical knowledge that they already possess, in problem solving situations.

Reference

Foong, P.Y. (1993). Development of a Framework for Analysing Mathematical Problem Solving Behaviours. *Singapore Journal* of *Education, 1993.* Vol. 13, No. 1, pp. 61-75.

Polya, G. (1957). How To *Solve It.* (2nd Ed). New York: Anchor Books.