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HOW PROSPECTIVE MATHEMATICS TEACHERS SOLVE THE EQUATION $F(X) = F^{-1}(X)$ USING THE GRAPHING CALCULATORS

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Anecdotal evidence suggests that many pre-university students (Years 11 and 12) and some mathematics teachers in Singapore have several misconceptions in solving equations of the form $f(x) = f^{1}(x)$ and that the appropriate use of the graphing calculator (GC) has the potential to correct these misconceptions. This paper analyses a group of prospective teachers' responses to a diagnostic test and pre- and post-diagnostic test surveys which not only revealed various inappropriate uses of the GC in solving the equation $f(x) = f^{-1}(x)$ but also illustrated pedagogical roles the GC could play in correcting students' misconceptions about functions and the understanding of the concepts of functions. The paper also discusses the implications of these findings on classroom practices and pedagogical strategies pertaining to the use of the GC at the pre-university level.

Key words : Graphing Calculator, Functions, Pre-service Teacher Training

INTRODUCTION AND LITERATURE REVIEW

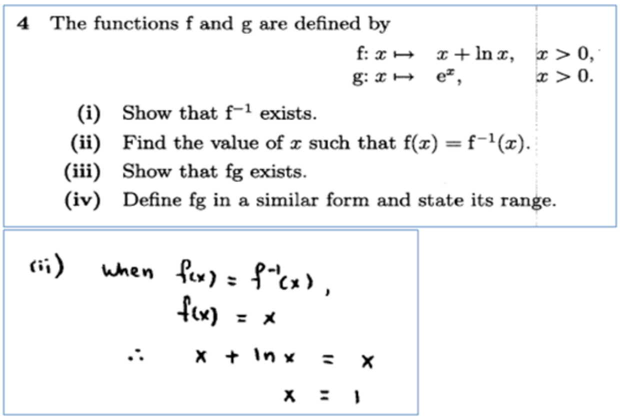
Functions and Graphs is one of the most important and popular mathematical topics in many mathematics curricula around the world. In the Singapore Mathematics curriculum, a preliminary concept of functions in the form of a straight line graph $y = mx + c$ is introduced at the lower secondary level (Years 7 and 8). Students proceed to learn more "complex functions" such as $y = x^2$ and $y = \sqrt{x}$ at the upper secondary level (Years 9 and 10). At the pre-university or Junior College (JC) level (Years 11 and 12), concepts such as the domain, range of a function, one-one functions, the existence of an inverse function and compositions of two or more functions are included in the JC Mathematics syllabus. The inclusion of the inverse and composite functions at the JC level gives students a more in-depth understanding of functions – an important mathematical construct a good conceptual understanding of which is essential and beneficial for students to pursue advanced Mathematics courses at the university level.

Over the years, there is a growing body of research that focuses on students' understanding and learning about functions and it was found that the concept of functions and the various representations of functions is difficult for many students (Leinhardt, Zaslavsky & Stein, 1990). With an increasing and irreversible trend of incorporating technology into teaching and teaching of mathematics in schools, many researchers have conducted studies on the use of technology in the teaching and learning of mathematics at all levels, in particular, the use of a graphing calculator (GC) on teaching and learning of functions and graphs has become an important area of research. A number of studies have shown that using a GC in graphing helps students to visualise and establish links between the different representations (numeric, algebraic and graphical) of a function (Ruthven, 1990, Borba & Confrey, 1996). However, there were fewer studies on the relation between teachers' content knowledge and pedagogical strategies of using GCs on the teaching of mathematics (Ellington, 2003), let alone studies on how prospective teachers' knowledge of using GCs in teaching of mathematics at the JC level.

BACKGROUND AND METHODTHLOGY

In Singapore, the Postgraduate Diploma in Education (PGDE) programme conducted at the National Institute of Education (NIE) is a one-year full-time teacher training programme aimed at preparing university graduates to become primary, secondary school or JC teachers. In particular, student teachers enrolled in the PGDE (JC) programme are trained to become JC teachers. As the use of a GC is integrated into the JC mathematics curriculum in Singapore and allowed in all formal assessments including school and national examinations, prospective teachers in the PGDE (JC) programme went through at least 12 hours of curriculum time on how to teach various JC Mathematics topics using a GC, one of which is Functions and Graphs.

Anecdotal evidence suggests that many JC students and some JC mathematics teachers have misconceptions in solving equations of the form $f(x) = f^{-1}(x)$ (Ng & Ho, 2012). For example, Figure 1 shows an examination question and a partial solution provided by the setter.



4 The functions f and g are defined by

$f: x \mapsto x + \ln x, \quad x > 0,$
$g: x \mapsto e^x, \quad x > 0.$

(i) Show that f^{-1} exists.
(ii) Find the value of x such that $f(x) = f^{-1}(x)$.
(iii) Show that fg exists.
(iv) Define fg in a similar form and state its range.

(ii) when $f(x) = f^{-1}(x)$,
$f(x) = x$
$\therefore \quad x + \ln x = x$
$x = 1$

Figure 1 : An examination question and its partial solution

The answer shown in Figure 1 is correct but the solution lacks mathematical rigour as there is no justification that solving the equation $f(x) = x$ is legitimate here. We feel that in order to prepare our prospective teachers adequately for their teaching career in a JC, it is of paramount importance that not only must the prospective teachers have a good conceptual understanding of functions and graphs, they must also know how to capitalise on the strengths of the GC and adopt appropriate pedagogical strategies in classroom teaching and in identifying and correcting students' misconceptions. We therefore carried out an exploratory study to ascertain whether

(a) prospective JC teachers have any misconceptions regarding solving the equation $f(x) = f^{-1}(x)$.

(b) a GC-based diagnostic test is able to elicit misconceptions and make prospective JC teachers aware of their misconceptions.

The study involved a class of 16 prospective teachers (henceforth known as participants) enrolled in the PGDE(JC) programme in the Year 2010. We highlight here that the participants were neither told that $f(x) = x$ is a sufficient but not necessary condition for $f(x) = f^{-1}(x)$ nor taught how to solve $f(x) = f^{-1}(x)$ prior to the study. With the intention of making the participants to use GCs, a set of six diagnostic test questions was specially crafted for the study. During a normal tutorial lesson, the class was given 20 minutes to respond to the six diagnostic test questions shortly after each participant had completed a pre-diagnostic test survey (henceforth known as pre-survey) consisting of eight items which are general statements pertaining to solving the equation $f(x) = f^{-1}(x)$. Participants were asked to circle either "Agree" or "Disagree" and give reasons. The objective of the pre-survey is to ascertain each participant's prior knowledge or method of solving the equation $f(x) = f^{-1}(x)$. The six different functions used in the diagnostic test are meant for "making" the participants aware of the different situations when the equation $f(x) = f^{-1}(x)$ has solutions. Immediately after the test, each participant answered a post-diagnostic survey (henceforth known as post-survey) which is identical to the pre-survey. The objective of conducting the pre and post-survey is to find out if the GC-based diagnostic test has any effect on influencing the participants' method of solving the equation $f(x) = f^{-1}(x)$.

RESULTS AND DISCUSSION

Diagnostic Test

The answer scripts of the diagnostic test were marked, compiled and analysed by the first author. The following tables show the summary of the results.

Table 1 : Summary of responses to Question 1

Question 1 : $f(x) = 2x + e^x$, for all real x.		
Participant's Method	Correct Answer	Wrong Answer
(a) Sketch the graphs of $y = f(x)$ and $y = x$ and solve $f(x) = x$	6	0
(b) Sketch the graphs of f and f^{-1} and solve $f(x) = x$	2	0
(c) Solve $ff(x) = x$	3	0
(d) Method not shown (answer obtained from GC)	4	0
(e) No solution – reason given : f^{-1} does not exist)	0	1

In Question 1, eight (50%) participants obtained the correct answer by solving $f(x) = x$ but none of them justified why this method works for this particular function. The function f is one-one and strictly increasing on the set of all real numbers. But, a closed form of its inverse function cannot be obtained. However, with the use of a GC (all the participants used TI84), the participants could "draw" the inverse function using the "Draw-inverse" command of the GC which enabled them to observe that there is a point of intersection between the graphs of f and f^{-1}. The fact that the x-coordinate of the point of intersection cannot be obtained from the GC in fact "stimulates" the participants to think further. From the answer scripts, we noticed that three students sketched the composite function $y = ff(x)$ and determined the solution to $ff(x) = x$. In this question, the "Draw-inverse" feature of the GC helps the participants to visualise that the graphs of f and f^{-1} being two different increasing curves which intersect each other, they must intersect on the line $y = x$. Indeed, the use of a GC here offers the participants a visual explanation of why solving $f(x) = x$ makes sense for this particular question.

Table 2 : Summary of responses to Question 2

Question 2 : $f(x) = \sqrt{7-3x}$, $x \le \frac{7}{3}$.			
Participant's Method	Correct Answer	Wrong Answer	Partially Correct
(a) Sketch the graphs of $y = f(x)$ and $y = x$ and solve $f(x) = x$	0	0	5
(b) Sketch the graphs of f and f^{-1} and solve $f(x) = f^{-1}(x)$	3	1	
(c) Solve $ff(x) = x$	2	1	2
(d) Method not shown	2		

As opposed to the function in Question 1, the function f in Question 2 is strictly decreasing. Less than 50% of the class (7 out of 16) obtained the correct answers. It was disappointing to see that five participants sketched the graph of f and the line $y = x$ before they solved $f(x) = x$ on a GC; these participants gave only one answer, missing out the other two answers. In fact, a good sketch of the graphs of f and f^{-1} on a GC, like what three other participants did, would have given them a clear illustration of the graphs intersecting at three different points. So, through this question and the use of a GC, the participants could visualise that if a function f is decreasing in its domain then the equation $f(x) = x$ has at most one solution and there may be other roots of the equation $f(x) = f^{-1}(x)$ that do not lie on the line $y = x$.

Question 3 is similar to Question 1 except that neither the graph of f nor the graph of f^{-1} intersects the line $y = x$. All the participants obtained the answer correctly which is "no solution". Once again, the graphs of f and f^{-1} displayed on the GC are useful for four participants to conclude that there is no solution. The rest of the participants either did not show the reason or gave the wrong reason as indicated in Table 3 below.

Table 3 : Summary of responses to Question 3

Question 3 : $f(x) = x + e^x$, for all real x.			
Participant's Method	Correct Answer	Wrong Answer	Wrong Reason Given
(a) Sketch the graphs of $y = f(x)$ and $y = x$ and solve $f(x) = x$	3	0	0
(b) Sketch the graphs of f and f^{1}	4	0	0
(c) Use the inequalities $x + e^x > x$ and $e^x > 0$ to conclude that $f(x) = x$ has no solution	0	0	3
(d) Method not shown	6	0	0

Table 4 : Summary of responses to Question 4

Question 4 : $f(x) = \dfrac{x-2}{x-1}$, $x \neq 1$.		
Participant's Method	Correct Answer	Wrong Answer
(a) Sketch the graphs of $y = f(x)$ and $y = x$.	3	5 (No solution)
(b) Sketch the graphs of f and f^{1}	4	1
(c) Solve $ff(x) = x$	1	0
(d) Method not shown	1	1

In Question 4, the function is deliberately chosen to be in the form $f(x) = \dfrac{x-a}{x-b}$ which is a self-inverse function when $b = 1$. From Table 4 above, we could see that about 56% (9 out of 16) of the participants obtained the answer correctly. Since the graph of f does not intersect the line $y = x$, using a GC to visualise the intersection points of the graphs of f and the line $y = x$ may give the participants an impression that there is no solution to the equation $f(x) = f^{1}(x)$. Indeed, out of eight participants who sketched the graphs of $y = f(x)$ and $y = x$ on their answer scripts, five of them gave "no solution" as their answers whereas the other three noticed the graph of f is symmetrical about the line $y = x$ and gave the correct answer. This question brings out an observation that if the participants just relied on the GC's output on the screen and did not think deeper to identify the symmetry property of the function, they would not be able to give the correct answer which is the set of all real numbers except $x = 1$. In addition, the use of a GC here illustrates quite clearly that the equations $f(x) = f^{1}(x)$ and $f(x) = x$ are not equivalent—this is clearly demonstrated by all self-inverse functions. Furthermore, the graphs of $y = ff(x)$ and $y = x$ being identical except at the point $x = 1$ should provide the participants with another useful insight on how a self-inverse function gives rise to the roots of the equation $f(x) = f^{1}(x)$.

Table 5 : Summary of responses to Question 5

Question 5 : $f(x) = 3 - x + e^{x-2}$, for all real x.		
Participant's Method	Correct Answer	Wrong Answer
(a) Sketch $y = f(x)$ and $y = x$.	0	4
(b) State that f^{-1} does not exist	12	0

In Question 5, as the function is not one-one thus the inverse function does not exist. Thus there is no solution to $f(x) = f^{-1}(x)$. However, four participants used the GC to solve the equation $f(x) = x$, obtaining two answers. The rationale for including this question is to highlight that the existence of a solution for $f(x) = x$ does not necessarily imply that there is solution for $f(x) = f^{-1}(x)$. This is the case in which a GC may mislead us easily if we do not examine the function carefully—an example of inappropriate use of a GC that our prospective teachers and students must be aware of.

Table 6 : Summary of responses to Question 6

Question 6 : $f(x) = \dfrac{x^3 - 1}{5x^3 - 1}$, for $x \neq \sqrt[3]{\dfrac{1}{5}}$.			
Participant's Method	Correct Answer	Wrong Answer	Partially Correct
(a) Sketch $y = f(x)$ and $y = x$.	0	4	0
(b) Sketch graphs of f and f^{-1}	3	2	1
(c) Solve $ff(x) = x$	5	0	0
(d) Sketch $y = f(x)$ and conclude that there is no answer	0	1	0

It was a disappointing performance for Question 6 (only 50% of the participants obtained all the answers). The function is not easy to sketch without the use of a GC. A careful examination of the graph of f should reveal that it is not symmetrical about the line $y = x$ but there are points on the graph which are symmetrical about the line $y = x$—thus these points satisfy the equation $f(x) = f^{-1}(x)$. Very clearly, by just sketching the graph of f and looking for its intersection with the line $y = x$ would result in no answer as seen by four of the participants. We were glad to see that eight participants could exploit the strength of a GC by either sketching both the graphs of f and f^{-1} or the graph of the composite function f^2 to obtain the correct answers.

Pre and Post-diagnostic Test Survey

Prior to the diagnostic test, we administered the pre-survey. We instructed the participants to respond to the survey as fast as possible so that the students' spontaneous responses to solving the equation $f(x) = f^{-1}(x)$ can be elicited—this is one way we sought to find out the pre-conceived or prior knowledge of the participants on solving the equation. Shortly after the survey, the above diagnostic test was administered and after which the students were given the same set of survey questions again. The following tables summarise the survey results.

In Table 7 below, the responses to Item 1 of the pre-survey revealed that five participants had the misconception of the equations $f(x) = f^{-1}(x)$ and $f(x) = x$ having the same solution set. These five participants provided the reasons that the graphs of f and its inverse f^{-1} are symmetrical about the line $y = x$ and thus they intersect on the line $y = x$. Out of the 11 participants who responded "Disagree", only two of them provided a valid example. The valid examples given were $f(x) = -x$ and $f(x) = 1/x$, $x \neq 0$. One participant reasoned that the solution set for $f(x) = x$ is a subset of $f(x) = f^{-1}(x)$ but he did not provide any example to explain his point. After the diagnostic test, four and five participants for Item 1 and Item 2 respectively switched from "Agree" to "Disagree" in the post-survey, suggesting that the diagnostic test may have enabled them to realise that the equations $f(x) = f^{-1}(x)$ and $f(\mathrm{x}) = x$ are not equivalent in general. However, the two participants who had converted from "Agree" to "Disagree" in fact gave a wrong reason in their survey form – one of them stated that " if $f(x) = x^2 - 1$, $x < 0$ then there are other solutions for $f(x) = f^{-1}(x)$ that do not lie on the line $y = x$" (in fact, as the domain of f does not include zero, this particular example has only solution lying on $y = x$).

Table 7 : Responses to the survey Items 1, 2 and 3

Survey Item	Pre-survey		Post-survey	
	Agree	Disagree	Agree	Disagree
1. The equation $f(x) = f^{-1}(x)$ and the equation $f(x) = x$ have the same solution set	5	11	1	15
2. I would always advise students to solve the equation $f(x) = x$ for the solutions of the equation $f(x) = f^{-1}(x)$.	7	8 (1 not sure)	2	14
3. Solutions of the equation $f(x) = f^{-1}(x)$ are the same as those which satisfy both the equations $f(x) = x$ and $f^{-1}(x) = x$.	5	8 3 (depend)	4 (1 depends)	11

From the responses to Item 2 of the survey in Table 7 above, we notice that there is an increase of six participants (from 8 to 14) who initially agreed with Item 2 now realised that roots of the equation $f(x) = f^{-1}(x)$ are not necessarily the same as those that satisfying both the equations $f(x) = x$ and $f^{-1}(x) = x$. The two participants whose response remained as "Agree" stated similar reasons that "solving $f(x) = x$ or $f^{-1}(x) = x$ was easier and a good starting point for solving the equation though it may not yield all the solutions". Conceptually, it is not correct and we are glad to be able to identify them and make them aware of the misconception.

For Item 3 in Table 7, we could also see that 11 out of 16 participants were certain that they would not always advise their students to solve $f(x) = x$ as compared to eight participants as reflected in the pre-diagnostic test survey.

Table 8 : Responses to the survey Items 4, 5 and 6

Survey Item	Pre-survey		Post-survey	
	Agree	Disagree	Agree	Disagree
4. A solution to the equation $f(x) = f^{-1}(x)$ always exist.	0	16	0	16
5. There are at most finitely many solutions to the equation $f(x) = f^{-1}(x)$.	1	15	0	16
6. The equation $f(x) = f^{-1}(x)$ and the equation $ff(x) = x$ have the same solution set.	7 (1 not sure)	8	10 (1 unsure)	5

The figures pertaining to Item 4 in Table 8 provides a clear evidence that the participants were aware that the equation $f(x) = f^{-1}(x)$ may not have any solutions. But it is rather disappointing that only four participants in the pre-survey were able to give a valid counter example to explain why they disagreed with the statement—3 participants cited $f(x) = e^x$ and another used a self-inverse function $f(x) = -x$. The rest of them wrote that "graphs of $y = f(x)$ and $y = x$ may not intersect" or "it all depends on the domain of the function". However, in the post-survey, more participants gave correct counter examples such as $f(x) = x + e^x$ (this is one of the diagnostic test questions). For Item 5, we are pleased to notice that after the diagnostic test, the only student who agreed that there are at most finitely many solutions to the equation $f(x) = f^{-1}(x)$ changed her mind and she wrote in the post-test survey that "f and f^{-1} might have the same graph". Indeed, our GC-based diagnostic test (in particular Question 3) could have enlightened her and enhanced her conceptual understanding that is related to Item 5.

For Item 6 in Table 8, we notice that three participants switched from "Disagree" to "Agree", reflecting that these participants have realised that the two equations $f(x) = f^{-1}(x)$ and $ff(x) = x$ are equivalent. For example, in Figure 2 below, Participant A worked out the composite function $y = ff(x)$ for Questions 4 and 6. He could write down the answer for Question 4 immediately as he noticed the composite function $y = ff(x)$ was $y = x$ whereas he used a GC to find the root of $ff(x) = x$ in Question 6. Thus, he demonstrated an understanding that solving $ff(x) = x$ is one valid alternative of solving $f(x) = f^{-1}(x)$.

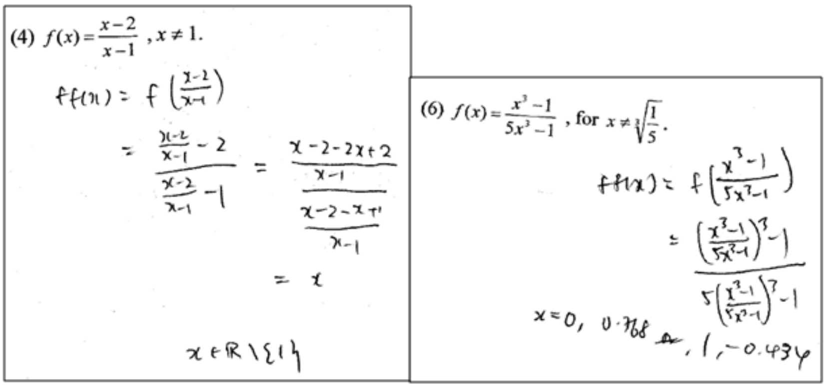


Figure 2 : Participant A found $y = ff(x)$ in Question 4 and Question 6

Table 9 : Responses to the survey Items 7 and 8

Survey Item	Pre-survey		Post-survey	
	Agree	Disagree	Agree	Disagree
7. I would always advise students to solve the equation $ff(x) = x$ for the solutions of the equation $f(x) = f^{-1}(x)$.	3	9 (4 not sure)	7	5 (4 not sure)
8. One way of solving the equation $f(x) = f^{-1}(x)$ is to find all a and b which satisfy the equations $f(a) = b$ and $f(b) = a$.	10	2 (4 not sure)	10	1 (5 not sure)

Item 7 in Table 9 above revealed that there was an increase of four participants that would advise their students to solve the equation $ff(x) = x$—Participant A was one of them. The reason given by these participants was that the inverse function $y = f^{-1}(x)$ may not be found easily or explicitly (for example in Question 1) whereas $y = ff(x)$ may be algebraically "complicated" but its graph can be sketched rather easily on a GC— most GCs can sketch the composite function $y = ff(x)$ once the function f is "keyed in". From the pre- and post-survey forms, we were able to identify those participants who responded with "not sure" and appropriate follow-up action can then be taken to help these participants.

Responses to Item 8 seem to suggest that the diagnostic test was not able to "convert" the six participants who either circle "Disagree" or wrote down "not sure" with regard to the truth of the item. However, the two surveys once again identified the participants who were doubtful of the statement. In fact, Questions 2 and 6 can be used to verify the statement (for example, in Question 6, $f(0.768) = -0.434$ and $f(-0.434) = 0.768$).

IMPLICATIONS AND CONCLUSION

In this paper, we described how a GC-based diagnostic test was designed to tease out the participants' misconceptions and their preferred methods of solving the equation $f(x) = f^{-1}(x)$. Our study shows that some participants had the tendency to solve the above equation using $f(x) = x$—a common method shown in Questions 2, 4 and 6 of the diagnostic test. From the responses to the post-survey, we notice that the diagnostic test was able to raise the participants' awareness that $f(x) = x$ is necessary but not sufficient for the equation $f(x) = f^{-1}(x)$ to hold and the existence of other valid methods of solving the equation $f(x) = f^{-1}(x)$. One implication of our findings is that JC teachers could use or adapt our diagnostic test questions as "teaching materials" to conduct the lesson on solving the equation $f(x) = f^{-1}(x)$. Furthermore, the different functions that we have chosen in the diagnostic test should be useful for JC students to discover that a self-inverse function and a function with both points (a, b) and (b, a) lying on its graph may have roots (for $f(x) = f^{-1}(x)$) that do not lie on the line $y = x$. It is important for students to understand how the behaviour and the property of the graphs of a function and its inverse function could give rise to the different approaches to solving the equation $f(x) = f^{-1}(x)$. Indeed, in order to help students to develop a robust conception of function and have a good mastery of content knowledge in functions, a more conceptual orientation to teaching the inverse and composite functions is one of the recommendations advocated by some university lecturers (Oehrtman, Carlson & Thompson, 2008). In addition, this study also shows that the graphing functionalities of a GC not only speed up the process of investigating and examining the graph of a function but also provide ideas for teachers and students to delve deeper, thus creating a possibility of discovering other class of functions for which the equation $f(x) = f^{-1}(x)$ has solutions that do not lie on the line $y = x$. Despite the limitation of a small sample size used for the study, we believe that this exploratory work has demonstrated a potentially useful pedagogical strategy of using the GC to facilitate students realising and correcting their own mathematical misconceptions and thus helping them to expand their knowledge further.

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