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The why, what and how of the 'Model' method: A tool for representing and visualising relationships when solving whole number arithmetic word problems

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Abstract:
The goal of this paper is to present a succinct consolidation of the why, what and how of the 'Model' method as there is a lack of such documentation for researchers, practitioners and policy makers thus far. The 'Model' method is a tool for representing and visualising relationships when solving whole number arithmetic (WNA) word problems. The paper traces the history of the method in the Singapore school mathematics curriculum, exemplifies the three basic models of the method, explores its efficacy and assesses its role as a problem solving heuristic in the Singapore primary school mathematics curriculum. The findings show that the method, a tool for representing and visualising relationships between quantities when solving whole number arithmetic word problems, was introduced in 1983 as part of the primary school mathematics curriculum for schools in Singapore. The method comprises three basic models, namely the part-whole model, the comparison model and the change model. Past studies carried out both in Singapore and other parts of the world, related to the use of the method and its impact on student learning show that the method does help students, including those with mathematical difficulties, improve in their ability to solve whole number arithmetic word problems. Perceptions of expert primary school mathematics teachers show that they engage students in solving whole number arithmetic word problems through Polya's framework and also Newman's strategies. They encourage students to represent on diagrams the givens and goals of a problem as this is a necessary step in visualising and recognising relationships. To do so they may use the method. Students may also use any other heuristic to solve the whole number arithmetic word problems.

Keywords: model method, whole number arithmetic, word problems, mathematical problem solving, primary school mathematics, Singapore

1. Background

A study carried out by the Singapore Ministry of Education (MOE) personnel in 1975 revealed that 25% of students after six years of primary school failed to meet the minimum numeracy level set by the ministry (MOE 2009). This and similar findings for other school subjects prompted the Prime Minister in August 1978 to call for a review of the education system that resulted in formulation of a New Education System (NES) (MOE 1979). The NES was implemented in 1981. The goal of the NES was to provide improved education for every child in the system.

As part of the NES the establishment of the Curriculum Development Institute of Singapore (CDIS) in June 1980 was an important milestone. Its main function was development of curriculum and teaching materials. It was directly involved in the implementation of syllabuses and systematic collection of feedback at each stage of implementation so that subsequent revisions and refinements would be strategic (Ang and Yeoh 1990). Among the various project teams, the Primary Mathematics Project (PMP) team led by Dr Kho was tasked to produce instructional materials for the teaching and learning of primary mathematics with effective teaching approaches and professional development of teachers (MOE 2009). In 1981,

the team administered diagnostic tests of basic skills of mathematics to a sample of 17000 Primary 1 to Primary 4 students. The findings were dismal with more than half of Primary 3 and 4 students doing poorly on items that tested division, 87% of Primary 2 to 4 students could solve word problems when key words like 'altogether' and 'left' were provided but only 46% of them could solve word problems without key words (MOE 1981).

At the CDIS Dr Kho and his team of curriculum writers, who were experienced teachers from schools and the Ministry of Education, together with expertise of international consultants, produced the new primary mathematics curriculum in 1981. The curriculum adopted a Concrete-Pictorial-Abstract approach for the teaching and learning of mathematics. This approach provides students with the necessary learning experiences and meaningful contexts, using concrete manipulatives and pictorial representations to construct abstract mathematical knowledge (CDIS 1987; Yip and Sim 1990; Ang 2008). As part of the new primary mathematics curriculum, the 'Model' method was introduced as a heuristic to help students solve word problems.

The goal of this paper is to present a succinct consolidation of the why, what and how of the "Model' method, as there is a lack of such documentation for researchers, practitioners and policy makers thus far. The paper is guided by three questions, which are as follows:

1. What is the 'Model' method?
2. What does research say about the efficacy of the 'Model' method?
3. What are the perceptions of expert primary school mathematics teachers about the 'Model' method as a heuristic for solving whole number arithmetic word problems?

2. What is the 'Model' method?

2.1 History of the 'Model' method in the Singapore primary school mathematics curriculum

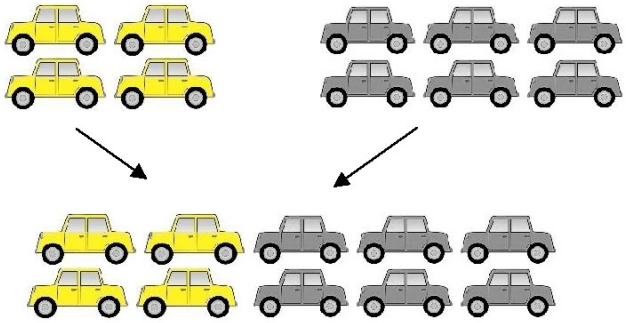
The only document that has a rather complete narrative of the 'Model' method is the monograph entitled "The Singapore model method for learning mathematics" (MOE 2009), It is reported in the monograph that to help students overcome difficulties they were having with word problems, Dr Kho and his team in 1983 innovated a method, the 'Model' method. The monograph does not state any research or curricula that guided the innovation. Research has shown that in the domain of word problems in the primary level, pupils face difficulties in grasping the problem structure and visualising the relationships between the quantities in the problems (Mason 2018). Furthermore, this innovation appears to have its roots in the Davydov curriculum (Schmittau and Morris 2004; Schmittau 2005; Bass 2015) which advocates algebraic thinking in the elementary grades. The Davydov curriculum engages students in analysing and identifying the structure of problems. In so doing students identify part-whole relations and find the missing part. It also resonates with the additive structure of Chinese mathematics education. When solving word problems, the additive structure focuses students on the arithmetic relationships between the parts and whole rather than the operator/s to compute the solutions (Mellone and Ramploud 2015; Sun and Bartolini Bussi 2018).

2.2 The three models

It was in 1987, that the first paper on the 'Model' method was presented, by Kho (1987). The paper stated the main features of the method, i.e., mathematical models in the form of pictorial representations, comprising three distinct models and illustrated with examples of how the models could be used to solve word problems. It did not provide any theoretical considerations that guided the method. It is apparent that the concrete-pictorial-abstract approach pervaded the design of the materials and pedagogy of teaching primary mathematics. This is congruent to Jerome Bruner's three stages of development, the enactive stage, iconic stage and symbolic stage (Bruner 1973). As explained in the Monograph (MOE 2009), the method was developed as a distinct intermediary activity that would activate the iconic phase of learning as shown in Figure 1.

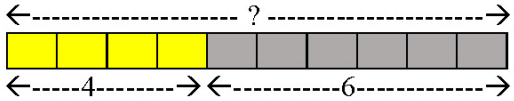
Consider the following problem
David has 4 toy cars. Ali has 6 toy cars. How many toy cars do they have altogether?

In Primary 1, students use concrete objects (or picture cut-outs) to form two groups of toy cars, and put the two groups together:

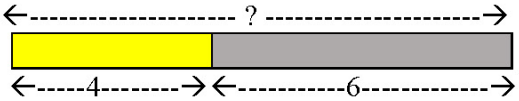


They add the two numbers 4 and 6 to give the total number.
Then they write the arithmetic equation 4 + 6 = 10 to solve the word problem.

In Primary 2, students may draw a pictorial model to represent the problem situation, e.g.



The model can be visualised as a whole comprising two parts. Students add the two parts to find the whole:



4 + 6 = 10
They have 10 cars altogether.

Fig. 1. The iconic phase of learning

The work of Zoltan Dienes also appears to provide a theoretical base for the method as it asks students to construct diagrams or make representations of word problems (Dienes 1971). As the

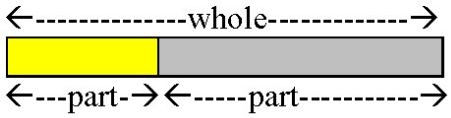
method uses a structured process whereby students are taught to visualise abstract mathematical relationships and their varying problem structures through pictorial representations (Ferrucci et al. 2008) it is a form of schema-based instruction (SBI) that teaches word-problem solving. SBI integrates schema theory with the effectiveness of relational diagrams. The strategy centres on the mathematical structure found in word problems as critical to problem comprehension and representation (Leh 2011). As in most SBI approaches, students are taught how to recognise word problems as falling into one of the several types (schema). Next, particular information from the context is added to the diagram. Using the diagram, the student then chooses a procedure for calculation. Once comprehension and representation are secure, students are more likely to choose correctly an operation and apply their procedural knowledge to a given word problem (Powell 2011).

The 'Model' method adopts steps that are similar to the SBI approaches. It leads students to draw rectangular "bars" or rectangles representing the quantitative parts and / or whole as given in the story context. The drawings are not simple pictorial renderings. They are schematic in nature as they refer to internal schema for a variety of problem types found in word problems (Foong 2010). The models do not depict actual objects from the word problem, but depict quantities and relationships between quantities (Ng and Lee 2009). Representations similar to that of the 'Model' method are also prevalent in other elementary school curricula. In Japan they are called tape diagrams and in the US "strip" diagrams (Murata 2008).

The 'Model' method helps students visualise the abstract mathematical relationships and the varying problem structures through pictorial representations (Kho 1987). The method was first introduced to primary four students in 1983 through the Primary Mathematics 4A textbook. An essential feature of the Singapore primary mathematics curriculum, the 'Model' method is now introduced to students in primary 1 (MOE 2009). In the following, we exemplify the three models that were created by Dr Kho and his team to support students in solving word problems involving whole number arithmetic. The models are the part-whole model, the comparison model and the change model. Each of these models serve a distinct purpose and the following examples illuminate it.

2.2.1 The part-whole model

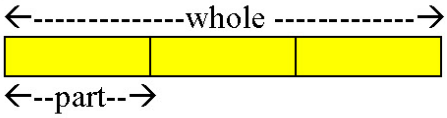
In the part-whole model (also sometimes referred to as the part-part-whole model), there is a quantitative relationship among the three quantities: the whole and the parts. This model helps students work through word-problems that involve relationships between the whole and its parts. Figure 2 shows the relationships.



To find the whole given two parts, students add:
part + part = whole

When the whole and one part is given, to find the other part, students subtract:
whole – part = part

A whole can also be divided into a number of equal parts



To find the whole given one part and the number of parts, students multiply:
one part x number of parts = whole

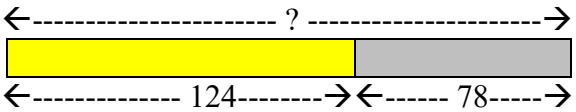
To find one part given the whole and the number of parts, students divide:
whole ÷ number of parts = one part

To find the number of parts given the whole and one part, students divide:
whole ÷ one part = number of parts

Fig. 2. Part-whole models

Examples, 1A to IE, illustrate the use of the part-whole models to solve word problems.

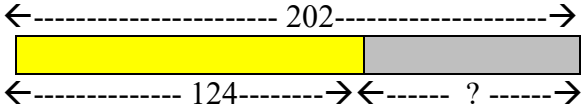
Example 1A
124 boys and 78 girls took part in a robotics competition. How many children took part in the competition?



A rectangular bar is drawn to represent the whole (total number of children). It is divided into two parts, representing the number of boys (124) and the number of girls (78) respectively. From the model, students can find the whole (shown by the symbol ?) by adding the two parts.

124 + 78 = 202
202 children took part in the robotics competition.

Example 1B
202 children took part in a robotics competition. If there were 124 boys, how many girls were there?

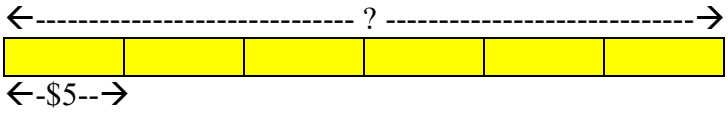


A rectangular bar is drawn to represent the whole (total number of children = 202). It is divided into two parts, representing the number of boys (124) and the number of girls (?) respectively.

From the model, students can find the part (shown by the symbol ?) by subtracting the given part (124) from the whole (202).

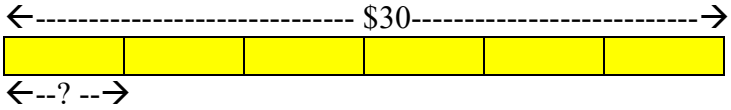
202 – 124 = 78
78 girls took part in the robotics competition.

Example 1C
6 children shared the cost of a gift equally. Each of them paid $5. What was the cost of the gift?



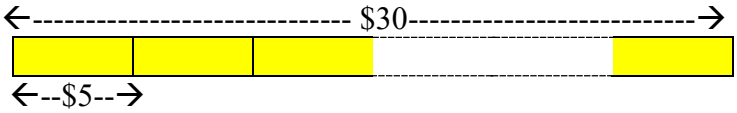
A rectangular bar is drawn to represent the whole (cost of the gift). It is divided into 6 equal parts, each representing $5. From the model, students can find the whole by multiplying one part ($5) and the number of parts (6).
6 x 5 = 30
The cost of the gift was $30.

Example 1D
6 children bought a gift for $30. They shared the cost equally. How much did each child pay?



A rectangular bar is drawn to represent the whole (cost of the gift = $30). It is divided into 6 equal parts, each representing?. From the model, students can find one part by dividing the whole ($30) by the number of parts (6).
30 ÷ 6 = 5
Each child paid $5.

Example 1E
A group of children bought a gift for $30. They paid $5 each. How many children were there in the group?



A rectangular bar is drawn to represent the whole (cost of the gift = $30). The broken part of the bar indicates that the number of parts is unknown. Knowing the whole and one part ($5), students can find the number of parts by dividing the whole ($30) by one part ($5).
30 ÷ 5 = 6
There were 6 children in the group.

2.2.2 The comparison model

The comparison model is used to compare two quantities to show how much one quantity is greater (or smaller) than the other. Without the model, students may simply rely on the cue words "more than" and use addition to solve the problem without realising that it is incorrect. Figure 3 shows the quantitative relationships among the three quantities: larger quantity, smaller quantity and the difference.

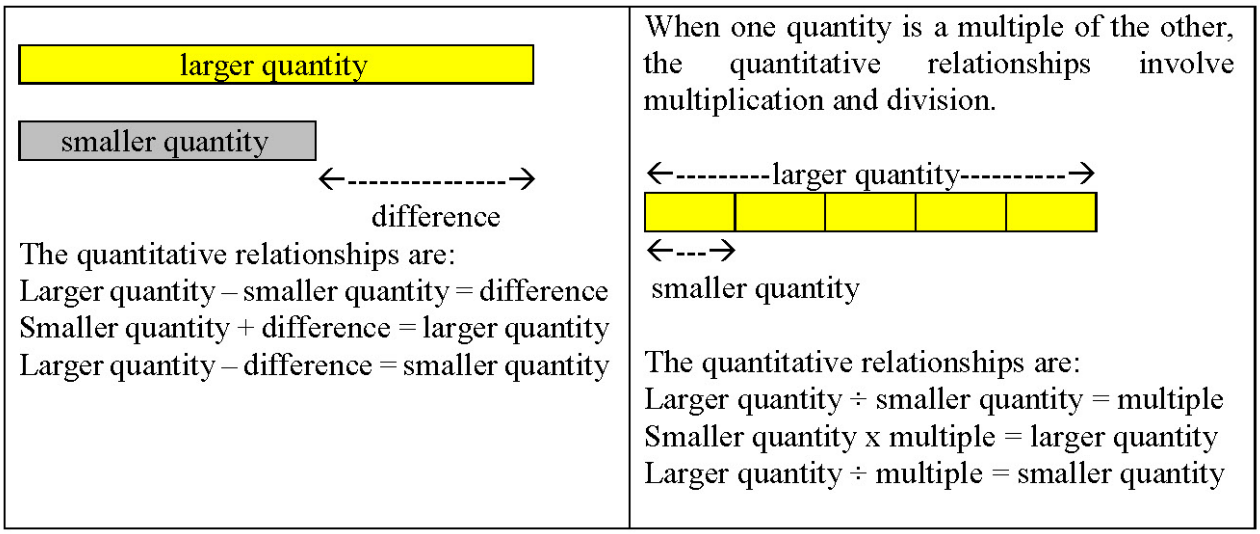
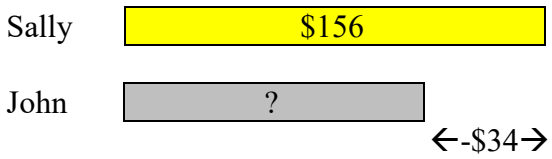


Fig. 3. Comparison models

Examples, 2A to 2F, illustrate the use of the comparison models.

Example 2A

Sally saved $156. John saved $34 less than Sally. How much did John save?

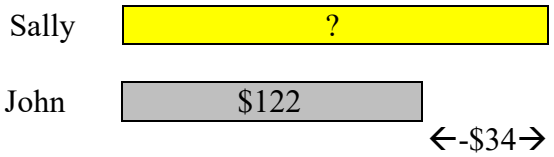


Two bars are drawn to represent Sally's savings and John's savings. In the comparison model, one bar is longer than the other, showing that Sally saved more than John. From the model, students can find the smaller quantity by subtracting the difference ($34) from the larger quantity ($156).

156 – 34 = 122

John saved $122.

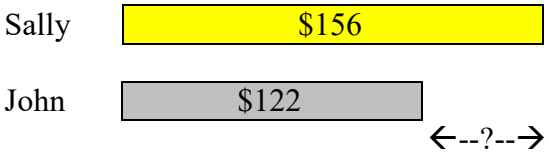
Example 2B
John saved $122. He saved $34 less than Sally. How much did Sally save?



In this example the smaller quantity (John's savings) and the difference are given. The larger quantity (Sally's savings) can be found by adding the smaller quantity ($122) and the difference ($34).

122 + 34 = 156
Sally saved $156.

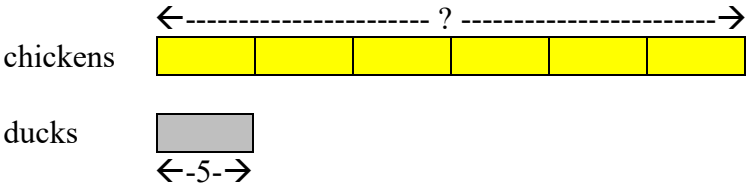
Example 2C
Sally saved $156. John saved $122. How much less than Sally did John save?



In this example, both quantities are given. Subtracting the smaller quantity ($122) from the larger quantity ($156) gives the difference.

156 – 122 = 34
John saved $34 less than Sally.

Example 2D
A farmer has 5 ducks. He has 6 times as many chickens as ducks. How many chickens does the farmer have?

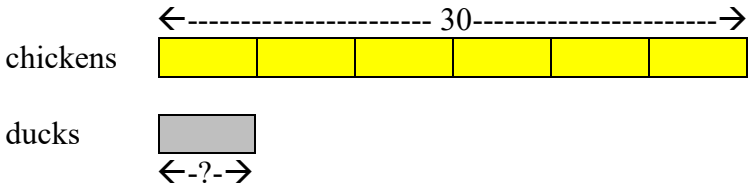


Two bars of lengths 6 units and 1 unit are drawn to represent the number of chickens and the number of ducks respectively. The model shows that the number of chickens is 6 times the number of ducks. Given the number of ducks (1 unit = 5), students can find the number of chickens (6 units) by multiplication.

6 x 5 = 30
The farmer has 30 chickens.

Example 2E

A farmer has 30 chickens. He has 6 times as many chickens as ducks. How many ducks does the farmer have?



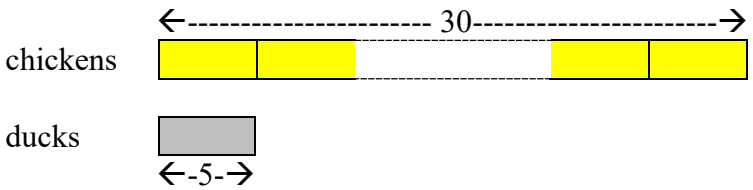
In this example, given the number of chickens (6 units = 30), students can find the number of ducks (1 unit) by division.

$30 \div 6 = 5$

The farmer has 5 ducks.

Example 2F

A farmer has 5 ducks and 30 chickens. How many times as many chickens as ducks does the farmer have?



The broken section in the longer bar indicates that the number of units is unknown. Given the two quantities, students can find the multiple by division.

$30 \div 5 = 6$

The farmer has 6 times as many chickens as ducks.

2.2.3 The change model

The change model shows the relationship between the new value of a quantity and its original value after an increase or a decrease. Knowing the increase or decrease, we can find the new value from the original value and vice-versa. This model is useful when solving word problems involving ratios of two quantities. Figure 4 shows change models.

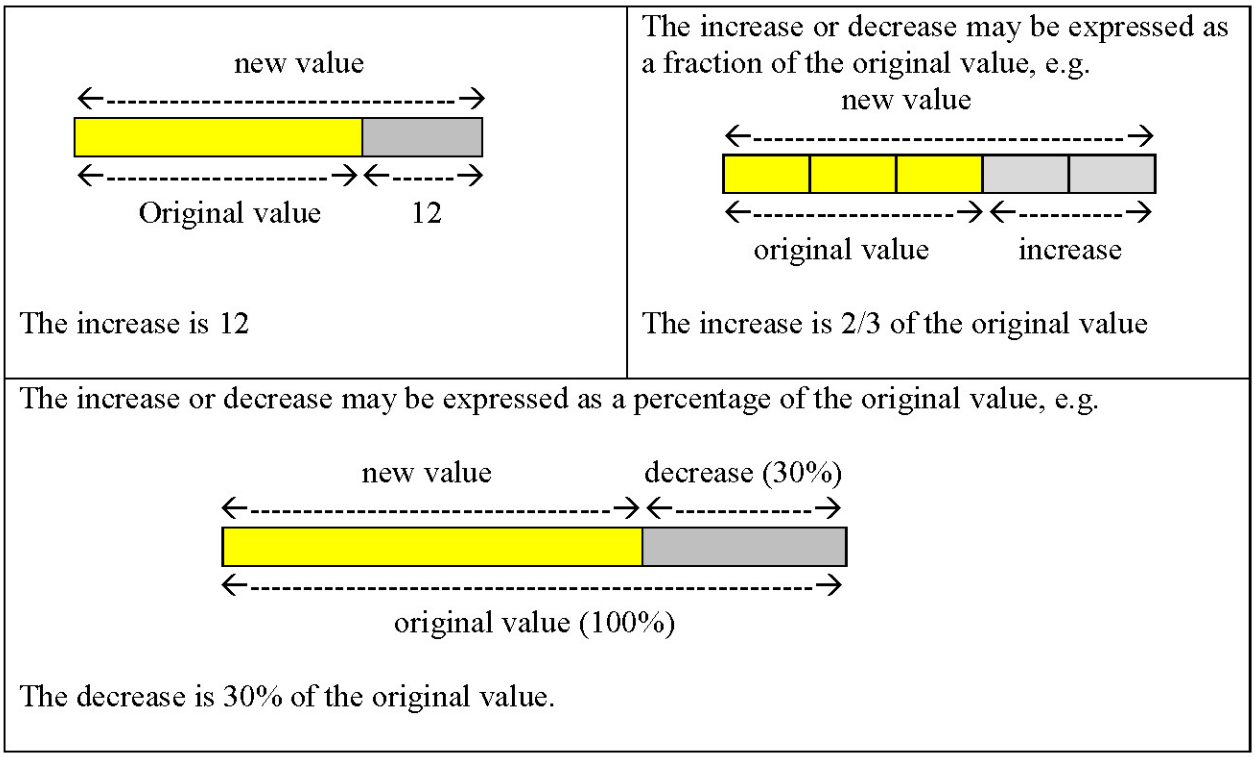
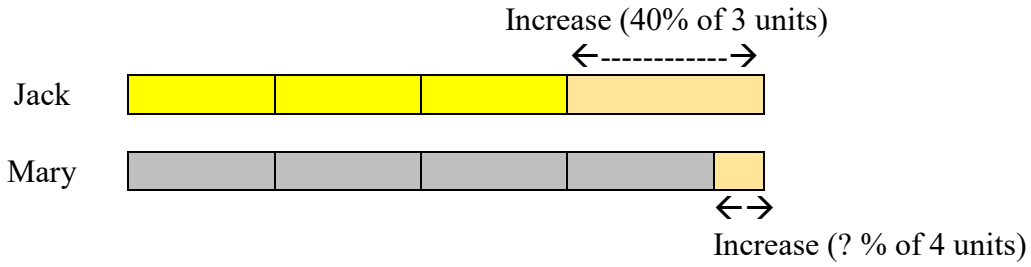


Fig. 4. Change models

Example 3 illustrates the use of the change model when solving a challenging word problem.

Example 3

Jack's salary and Mary's salary are in the ratio 3:4. If Jack's salary is increased by 40%, by what percentage must Mary's salary be increased or decreased so that they will have the same salary?



The model shows that Jack's salary is 3 units and Mary's salary is 4 units.
It also shows that their new salaries are equal.

Increase in Jack's salary = 40% x 3 units = 1.2 units
Increase in Mary's salary = 1.2 units – 1 unit = 0.2 unit
Percentage increase = $\frac{0.2}{4}$ x 100% = 5 %

2.3 The stencil for drawing models

As part of school stationary, stencils like the one shown in Figure 5 are available for students to use during mathematics lessons when they need to draw bar models.

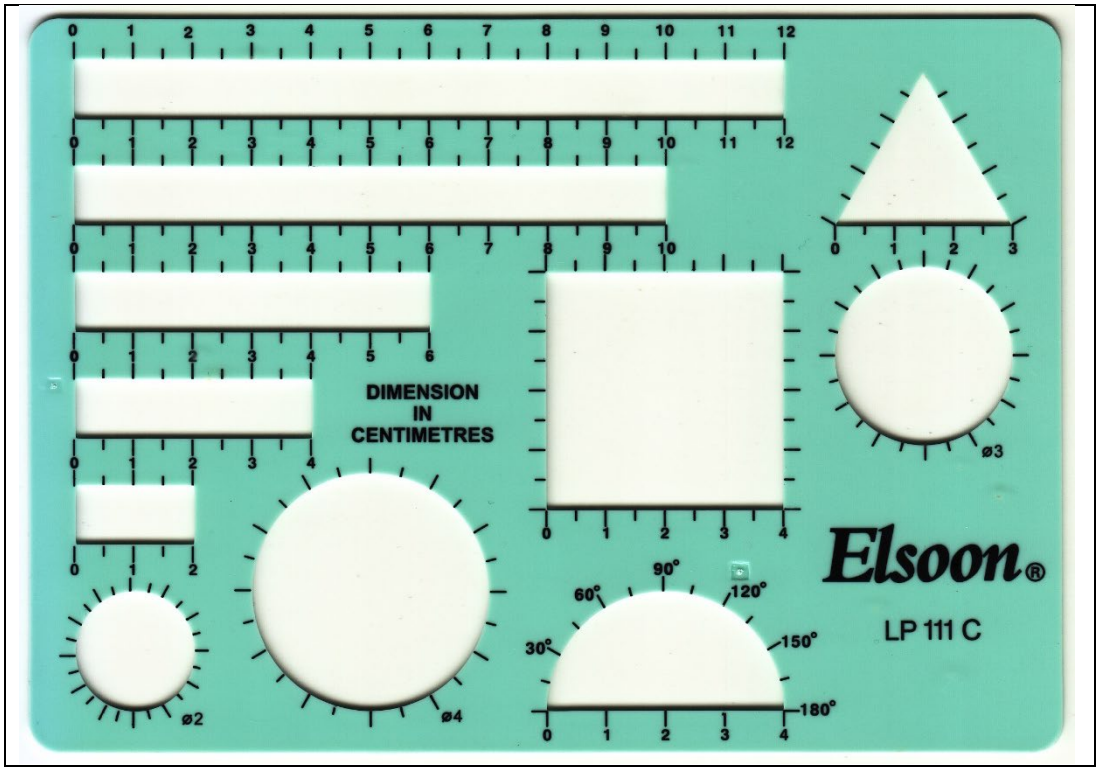


Fig. 5. A stencil for drawing bar models

3.0 What does research says about the efficacy of the 'Model' method?

The 'Model' method has been part of the Singapore primary school mathematics curriculum for more than three decades now. It is important to examine what research says about its efficacy. To do so, studies conducted in Singapore and also in other parts of the world that have adopted the method are reviewed in this section.

3.1 Research studies conducted in Singapore

In 1987 when Kho first introduced to the mathematics education community at large the 'Model' method he also expressed hope that "some studies can be carried out in the classrooms to determine the effectiveness of the method and to improve the instruction of the method (Kho 1987, p. 351).

However, a search for literature on the efficacy of the method and / or improvement of instruction of the method has revealed a dearth of such studies. The few studies carried out so far in Singapore, Ng and Lee (2005, 2009), Kaur et al. (2004), Goh (2009) and Poh (2007), on the method and involving students have generally shown that it does help students improve their ability in solving arithmetic word problems.

Ng and Lee (2005), in their study involving 151 primary 5 students on the use of the model method to solve word problems, found that pupils used the method most successfully with an arithmetic word-problem. However, when solving algebraic word-problems the rate of success decreased from word problems where the nature of relationships was homogenous involving two additive relationships to that which involved non-homogenous relationships. An arithmetic word-problem is one in which the relationships between the known and unknown states are 'connected' and can easily be found, while these relationships in algebraic word-problems are 'disconnected' (Bednarz and Janvier 1996). An example of an arithmetic word-problem (Ng and Lee 2005, p. 63) and an algebraic word-problem (Ng and Lee 2005, p. 63) each are shown in Figure 6.

Dunearn Primary school has 280 pupils. Sunshine Primary school has 89 pupils more than Dunearn Primary. Excellent Primary has 62 pupils more than Dunearn Primary. How many pupils are there altogether?	A cow weighs 150 kg more than a dog. A goat weighs 130 kg less than the cow. Altogether the three animals weigh 410 kg. What is the mass of the cow?
Arithmetic word-problem	Algebraic word-problem

Fig. 6. Word problems (Ng and Lee 2005, p. 63)

Ng and Lee (2005) also found that pupils who used the model method to solve word problems involving fractions demonstrated that they had a superior command of how they visualised problems. Their solutions showed that they oscillated between the symbolic representations of the problem and its visual analogue.

In another study involving 965 primary 5 students Kaur et al. (2004) found that students who used the model method were more successful in solving the word problems they were asked to solve as part of an assessment of their mathematics attainment in primary school. Figure 7 shows the problems and some data. It is also apparent from Figure 7 that students found problems 2 and 3 more challenging and so they resorted to drawing models and when they were able to construct an appropriate model they were successful in obtaining a correct answer.

Problem	Facility index	Heuristic used to obtain correct answer
Problem 1 Mr Lim sold 5 oven toasters and 3 rice cookers for $500. If a rice cooker cost $20 less than an oven toaster, find the cost of a rice cooker.	47.1 %	No model drawn—38.8% Correct model drawn—45.0% Incomplete / incorrect model drawn—16.3%
Problem 2 Devi has 3 times as much money as Sally. Mingfa has $50 less than Devi. If they have $230 altogether, how much money does Mingfa have?	38.6%	No model drawn—10.2% Correct model drawn—89.0% Incomplete / Incorrect model drawn—0.8%
Problem 3 Ali has $120 more than Siti. After Siti had spent $40, Ali had twice as much money as Siti. How much money did Ali have?	50.3%	No model drawn—13.1% Correct model drawn—85.3% Incomplete / Incorrect model drawn—1.6%

Fig. 7. Word problems (Kaur et al. 2004, pp. 43-47).

Goh (2009) in her study focused on examining errors 32 mixed ability primary 5 pupils made when solving complex relational word problems using the Model method. She found that pupils were able to solve 1-step word problems involving multiplicative ratio such as the following:

> Sally reads 30 books in a year. Rani reads three times as many books as Sally. How many books does Rani read in a year? (p. 93)

They used the Model method with ease. However, when solving multi-step and before-after type of problems such as the following, they faced difficulties:

> Danny had four times as much money as Peter. After Danny gave Peter $600, Peter had four times as much as money as Danny. How much money did they have altogether? (p. 93)

Errors made by the pupils were of two types, transformation and translation. Transformation errors included i) partial transformations where the value of change (either known or unknown) and the correct number of basic units showing the relationships between elements in the *after* changes were omitted and ii) non-integration of the values of changes to form a coherent resultant model. The translation errors included the inability to identify and calculate the value of one basic unit from correct models drawn, incorrect translations into number sentences based on incorrect models drawn, and incorrect direct translations into number sentences based on random cues.

Poh (2007) in her study, involving 341 grade 3 pupils at the beginning of their school year, found that pupils did not always draw bar models as part of the Model method to solve word problems. Some used alternative means and arrived at the solution via logical reasoning, particularly when the word problems involved one or two steps only. Some others had gaps in their understanding that hindered their ability to solve the word problems. The gaps included a) understanding the word problem, and b) mastery of the 'Model' method. Lack of mastery of the method led to inability to decide what the correct model should be, and given a model the inability to determine the steps leading to a correct solution.

Ng and Lee (2009) noted that all the 18 teachers in their study agreed "that the visual and concrete nature of the model method made it a useful problem-solving tool" (p. 297). The teachers also adopted the unitary method when demonstrating to their students how to solve for the unknown when solving the word problems. In so doing, students were expected only to create "a series of logical arithmetic equations" (p. 297). Ng and Lee in their study of 151 primary 5 students solving word problems using the Model method found that successful problem solvers drew detailed models and constructed accurate arithmetic equations to represent the information represented by the model drawing. More importantly the study showed that using the Model method is "not an all-or-nothing process" (p. 310), as one-third of the partially correct solutions were a consequence of "misinterpretation of information correctly captured in the model, misinterpretation of a piece of information, or changing the generator midway through the solution" (p. 310). The study also noted that "average ability

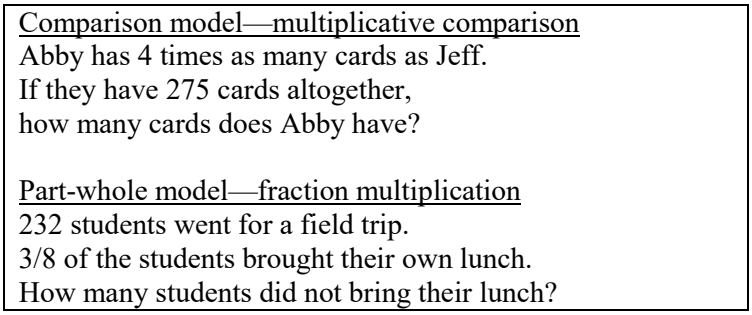
children's solution of word problems involving whole numbers could be improved if they learn to exercise more care in the construction of related models" (p. 311).

3.2 Research studies conducted in Pakistan, US, and Greece

The excellent achievement of Singapore's students in the Trends in International Mathematics and Science Study series from 1995 till 2015, and also Programme for International Student Assessment (PISA, 2009, 2012 and 2015) has drawn a lot of attention to school mathematics curricula, specifically to textbooks used in Singapore schools. In several countries, the 'Model' method has been emulated in classrooms. However, again there appears to be a dearth of research done on its efficacy in these classrooms. The few studies done so far, Jan and Rodrigues (2012), Mahoney (2012), Koleza (2015) and Morin et al. (2017) all reinforce efficacy of the 'Model' method.

Jan and Rodrigues (2012) introduced model drawing as a strategy to teach mathematical word problems at the upper elementary level to a group of students in Pakistan. Model drawing was used in accordance with Polya's problem solving approach (Polya 1957) and this resulted in the students better articulating their understanding of word problems. Students were also able to formulate mathematical equations with ease and confidence as the models they drew helped them visualise relationships between the knowns and unknowns of the word problems.

In the US, Mahoney (2012) examined the effect of the 'Model' method on word problem-solving of third and fourth grade students. Employing a single-case design, a researcher-designed teaching intervention was delivered to a child in third grade over the course of eight teaching sessions. Using researcher designed assessment probes, repeated measures of the dependent variable (percentage of problems solved correctly) were taken throughout the experiment through three different phases: baseline, intervention and maintenance. The design was repeated across four different third and fourth grade participants. The results demonstrated a positive functional relationship between the independent variable (the model drawing intervention) and the participant's problem solving performance. The percentage of problems solved correctly rose significantly as soon as the intervention phase began and the child employed the Model method in solving complex word problems. The pattern was repeated across two different types, multiplicative comparison word problems and fraction word problems (see Figure 8). The results showed a similar functional relationship between model drawing and word problem-solving performance.


Comparison model—multiplicative comparison
Abby has 4 times as many cards as Jeff.
If they have 275 cards altogether,
how many cards does Abby have?

Part-whole model—fraction multiplication
232 students went for a field trip.
3/8 of the students brought their own lunch.
How many students did not bring their lunch?

Fig. 8. Word problems (Mahoney 2012, pp. 131–132)

In the research study by Koleza (2015), 19 third graders in Greece were taught multiplication tables during eight 45 minutes sessions of regular lessons. Their lessons were enriched by the bar model as a way to connect multiplication and division in a unified scheme. They worked on tasks which were both arithmetic and algebraic in nature. When the teacher introduced complementary or inverse problems as the two variations of an initial multiplication problem (see Figure 9), "students faced two major difficulties: i) expressing in natural language the problem they had already represented by the bar model, and ii) representing by the bar model the quotative/measurement division problem variation" (p. 1943). Students found the multiplication problem and the inverse problem 1 (shown in Figure 9) easy to represent in the form of a bar model, but inverse problem 2 (see Figure 9) difficult as they had to 'deconstruct' the bar model. The study showed that the bar model is an effective model, and most students after a relatively short term of instruction were able to formulate and represent in the form of a bar model a multiplication and the corresponding partitive division problems. The study also warranted more research about the kind of representation that is more appropriate for the quantitative division problems.

Multiplication problem
If I save 8 euro in a week, how many euros I will have after five weeks?

Inverse problem 1
If I saved 40 euros in 5 weeks, how many euros did I save in a week?

Inverse problem 2
In how many weeks did I save 40 euros if I saved 8 euros each week?

Fig. 9. Word problems (Koleza 2015, p. 1943)

Students with mathematics difficulties (MD) find mathematics word problem solving especially challenging. Morin et al. (2017) studied the effects of a problem-solving strategy, bar model drawing, on the mathematical problem-solving skills of students with MD in the US. The study extended past research that suggests that schematic-based instruction and cognitive strategy instruction delivered within an explicit instruction framework can be effective in teaching various mathematics skills related to word problem solving. A multiple-baseline design replicated across groups was used to evaluate the effects of intervention of bar model drawing on mathematics problem-solving performance of students with MD. Student achievement was measured in terms of increased correct use of cognitive strategies and overall accuracy of mathematics word problem solving. Results showed that bar model drawing is an effective strategy for increasing elementary students' accuracy in solving mathematics word problems, involving one or more variables using addition, subtraction and multiplication, and their ability to use cognitive strategies to solve work problems. It is apparent that the bar model drawing was similar to the 'Model' method, which is a part of the Singapore school mathematics curriculum for primary schools. In Figure 10, the bar model drawing for each of the 8 types are matched with those of the 'Model' method. In addition, how the models would be drawn according to the 'Model' method are also shown. Therefore it may be inferred that the 'Model' method has also shown promise for students with mathematics difficulties (MD).

3.2 Discussion

Introduction of the 'Model' method in the Singapore school primary mathematics curriculum in the 1980s was an attempt to improve students' ability to solve word problems. Over a period of

more than three decades the method has facilitated a consistent and coherent use of a representation that bridges student understanding focussing on mathematical relationships and problem-solving process. A pedagogical tool, for visualising and representing relationships, it pervades instructional materials, such as textbooks approved by the Ministry of Education for use in schools (Kaur 2015).

Research, carried out on the efficacy of the method in Singapore, shows that it helps students solve arithmetic word problems more successfully than algebraic word problems (Ng and Lee 2005; Kaur et al. 2004). It is also noteworthy that the method facilitated the visualisation of relationships involving fractions in word problems (Ng and Lee 2005) and multiplicative ratios in one step word problems (Goh 2009). However, transformational and translation errors, do affect students' success when solving multi-step and before-after type of problems (Goh 2009). Furthermore, students do not always draw models to solve word problems when they have alternative ways to do so (Poh 2007). In the report of Ng and Lee (2009), teachers agreed "that the visual and concrete nature of the model method made it a useful problem-solving tool" (p. 297) but it was also "not an all-or-nothing process" (p. 310).

Research carried out elsewhere on the efficacy of the method is also positive. It has helped students to better articulate their understanding of word problems (Jan and Rodrigues 2012) and improve their problem-solving performance following use of the 'Model' method to solve complex word problems (Mahoney 2012). In addition, students, after a relatively short term of instruction in the 'Model' method were able to formulate and represent in the form of bar models, multiplication and the corresponding partitive division problems (Koleza 2015). Elementary students with mathematics difficulties have also benefitted from a bar model drawing strategy that is essentially the 'Model' method (Morin et al. 2017). The method increased their accuracy in solving word problems involving one or more variables using addition, subtraction and multiplication.

The efficacy of the 'Model' method is not surprising as it is rooted in part-whole relations. It focuses the student on perceiving part-whole relationships and becoming aware of structure, which Resnick et al. (1991) described as proto-quantitative part-whole schemas "that organise children's knowledge about the ways in which material around them comes apart and goes together" (p. 32). When attention is drawn to the development of part-whole relations, addition and subtraction no longer appear as separate operations but rather as dialectically interrelated actions that arise from the part-whole relation between quantities (Schmittau 2011, p. 77).

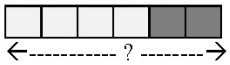
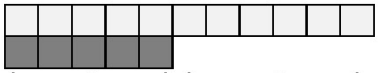
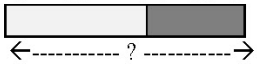
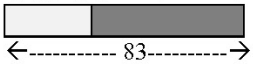
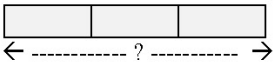
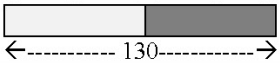
Lesson Number	Bar model drawing type	Operation(s)	Example of word problem
1	Discrete model with one variable Part-whole model [given the parts , find the whole]	addition	Olivia ate 3 cookies for lunch and 2 more cookies after dinner. How many cookies did she eat all together? ← 3 →← 2 →  ← ? →
2	Discrete model with one variable Part-whole model [given the whole , find the part]	subtraction	Sam had 9 pencils, but he gave 2 away. How many did he have left? ←---------- 9 ----------→  ← 2 → ←----------? ----------→
3	Discrete model with more than one variable Part-whole model [given the parts , find the whole]	addition	Jeannette saw 4 snakes and 2 frogs while she was hiking. How many amphibians did she see all together? ←--- 4 ------→← 2 →  ←---------- ? --------→
4	Discrete model with more than one variable Comparison model	subtraction	Ari had 11 basketballs, while Nicco had 5 footballs. How many balls did Ari have than Nicco? ←---------- 11 ----------→ Ari Nicco  ←------- 5 ------→←-------- ? -------→
5	Continuous model with one or more variables Part-whole model [given the parts , find the whole]	addition	Sarah had 53 fiction and 31 nonfiction books. How many books does Sarah have in all? ←----53-----→←--31-→  ←---------- ? ----------→
6	Continuous model with one or more variables Part-whole model [given the whole , find the part]	subtraction	Thomas had 83 M&Ms, but he dropped 29. How many M&Ms does he have now? ←-29-→←-----?-------→  ←---------- 83----------→
7	Discrete or continuous model with one or more variables Part-whole model [given the parts , find the whole]	multiplication	Avery, Jackson, and Hayden each have 32 baseballs cards. How many cards do they have in all? ← 32 →← 32 →← 32 →  ← ---------- ? ---------- →
8	Part-whole model with one or more variables	addition and subtraction	Paula had 130 antique buttons. 65 were made of metal, and the rest were glass. How many glass buttons does she have? ←----65-----→←-----?----→  ←---------- 130----------→

Fig. 10. Word problems (Morin et al. 2017, p. 94)

4. What are the perceptions of expert primary school mathematics teachers about the 'Model' method as a heuristic for solving whole number arithmetic word problems?

In this section, we present a small scale study that examines the perceptions of expert primary school mathematics teachers about the 'Model' method as a heuristic for solving whole number word problems. The study draws on a purposive sample, comprising expert primary school mathematics teachers. The choice of involving the expert teachers is deliberate. These expert teachers are the ones who impact the classroom pedagogy of other mathematics teachers in Singapore primary schools. Their professionalism also allows the study to tap in to the individual positions they take with regard to the enactment of the school mathematics curriculum, irrespective of approaches advanced by textbooks.

4.1 Participants and instrument

Altogether 5 teachers participated in the study. They are all experts by virtue of their responsibilities in developing the competence of fellow teachers in the teaching and learning of mathematics in primary schools. Two of them are principal master teachers with more than 30 years of teaching experience each. There are only two such primary mathematic teachers in Singapore. Another one is a master teacher with more than 20 years of teaching experience. Principal master teachers and master teachers are curriculum leaders at the Academy for Singapore. Their deep knowledge, wide experience and commitment to excellence drives the on-going professional development of teachers in Singapore. They provide support for teachers at the national level. The remaining two are a lead teacher and a senior teacher with more than 17 years of similar experience each. Lead teachers provide curriculum leadership at the zonal level while a senior teacher provides the same at the school level. Schools in Singapore belong to one of four zones, east, west, north or south.

As part of the study the teachers were interviewed by a mathematics educator from the National Institute of Education (NIE) in Singapore. NIE is the sole teacher education institute in Singapore. The mathematics educator has been at the institute for more than 30 years and is deeply involved in mathematics education of the nation. The interviews were guided by the prompts shown in Figure 11. The interviews were not audio or video recorded. Only hand written notes were made by the researcher, and some solutions sketched on paper by the participants were collected.

1. Presently, the concept of models is introduced intuitively in primary 1 and 2 and explicitly from primary 3 onwards. Do you feel this helps students in representing and visualizing relationships involved in both simple and complex word problems?
2. At which year level do students rely more on models to solve word problems and for which topics?
3. How would you guide your students in solving the following word problems? Also say at which year level should the students be able to work with such problems? Which heuristic/s would you suggest they use?

Problem 1

Bryan had 492 postcards at first. After giving Jean 165 postcards, he had 58 fewer postcards than Jean. How many postcards did Jean have at first?

Problem 2

Alice and Ben spend $65. Alice and Cindy spent $185. Cindy spent three times as much as Ben. How much did Alice spend?

Problem 3

David is 6 years old. Shawn is 30 years old. In how many years' time will Shawn's age be 4 times David's age?

Problem 4

Benedict used 84 sticks to make a total of 22 triangles and pentagons. A triangle was made using three sticks and a pentagon was made using 5 sticks. How many pentagons were there?

Problem 5

In a basket, there are apples and oranges. 25% of the fruit are oranges. Mary buys more oranges and puts them into the basket to double their number. What percentage of the fruit are apples after Mary doubles the number of oranges?

Fig. 11. Interview prompts

4.2 Data and findings

This section presents the data and findings chronologically in order of the prompts (shown in Figure 11).

Presently, the concept of models is introduced intuitively in primary 1 and 2 and explicitly from primary 3 onwards. Do you feel this helps students in representing and visualizing relationships involved in both simple and complex word problems?

All the five teachers agreed that the approach adopted in the curriculum materials, in particular the textbooks (approved for use in schools by the MOE), with regard to the introduction of the concepts of models in primary 1 and 2 (intuitively) and from primary 3 onwards (explicitly) was a good one. They felt the approach was very helpful is helping students visualize relationships, like addition "putting together the two groups of objects" and also the reverse. The use of concrete manipulatives and drawing parallels to bar diagrams such as those shown in Figure 1 seeded the development of an approach on which students can draw in the later year levels when the need arises. The teachers emphasized that the introduction to the 'Model' method is gradual and many students who are able to make the relationships between the given and goals of word problems, would write the mathematical sentences, such as 3 + 5 = ?, 8 – 3 = ?, etc, without drawing the respective bar models. However, when the word problems do get complex, they are pushed to draw bar models, represent the data and visualize relationships, and work towards a solution.

At which year level do students rely more on models to solve word problems and for which topics?

The teachers felt that for word problems that required students to make sense of 'more than' / 'less than' conditions, they are always pushed to draw bar models of the comparison type. Also for topics such as fractions, ratio, and percentages students find models to be very helpful. Generally, the teachers felt that from primary 4 onwards, students tend to draw bar models when solving complex word problems.

How would you guide your students in solving the following word problems? Also say at which year level should the students be able to work with such problems? Which heuristic/s would you suggest they use?

Problem 1
Bryan had 492 postcards at first. After giving Jean 165 postcards, he had 58 fewer postcards than Jean. How many postcards did Jean have at first?

All the teachers concurred that this is a complex word problem, but students in Primary 4 may be asked to find a solution for it. The teachers also felt that this complex word problem is challenging as it involves the use of the before-after concept. If they were guiding their students in finding the solution, they would do it as follows:

- Allow students time to read the problem / check for their understanding—ask students to explain what the givens and goal of the problem are.
- Engage students in acting out the problem / encourage students to discuss the problem in pairs / small groups.
- Ask the students to draw a diagram to represent the problem (will not explicitly ask them to draw bar models).
- Ask a few students to share their diagrams on the board of the class.
- Engage the whole class in examining the diagrams on the board / focus on what the unknown is—How many cards does Jean have? and how can we find it?
- Help students improve their bar models.
- Ask students to find "how many cards did Jean have at first?"
- Encourage them to share their answers and check for any errors they made.

All the teachers were hopeful that their students would after drawing a few iterations of models eventually draw one, as shown in Figure 12, depicting the "after giving Jean 165 cards" situation.

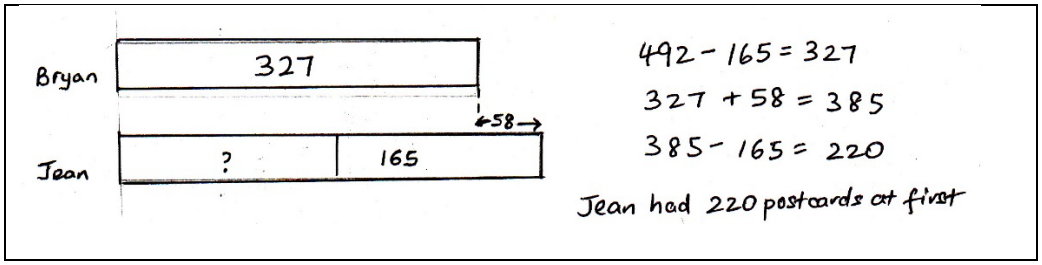


Fig. 12. Possible solution for problem 1

They also highlighted that their approach was guided by Polya's (1957) stages of problems solving, namely Read, Understand, Plan and Do, and Check. They were also cognizant that students may have difficulties at any of the following junctures:

- Reading—word and symbol recognition
- Comprehension—what are the givens and goal (what to find)
- Transformation—make a diagram / write number sentences
- Process skills—work out the sums
- Encode the solution

when solving the word problems (Newman 1983).

Problem 2
Alice and Ben spend $65. Alice and Cindy spent $185. Cindy spent three times as much as Ben. How much did Alice spend?

All the teachers agreed that this was a problem that students in primary 4, 5 or 6 would be able to work on using manipulatives and by making a representation. Again the teachers reiterated that they would take the students through the routine they use for engaging students in problem solving which is as follows:

- Allow students time to read the problem / check for their understanding—ask students to explain what the givens and goal of the problem are.
- Engage students in acting out the problem / encourage students to discuss the problem in pairs / small groups.
- Check that students see the sum of money spent by Alice is the same in both cases.
- Ask students to articulate the relationship between the sums of money spent by Ben and Cindy.
- Ask the students to draw a diagram to represent the problem (will not explicitly ask them to draw bar models).
- Ask a few students to share their diagrams on the board of the class.
- Engage the whole class in examining the diagrams on the board / focus on what the unknown is—How much did Alice spend? and how can we find it?
- Help students improve their diagrams.
- Ask students to find "how much did Alice spend?"
- Encourage them to share their answers and check for any errors they made.

The teachers were confident that not all the students would draw bar models, as shown in the right hand side of Figure 13. They said many would use counters as manipulatives and later draw a diagram like the one on the left side of Figure 13. Both representations would lead students to the same answer.

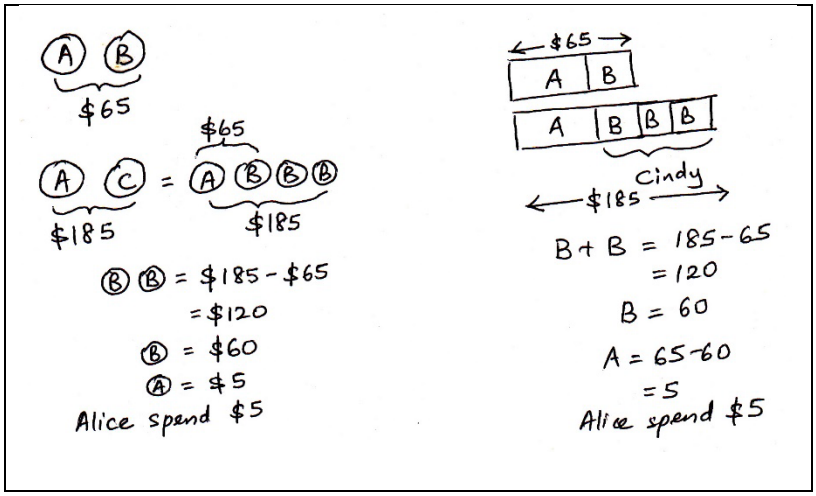


Fig. 13. Possible solutions for problem 2

Problem 3

David is 6 years old. Shawn is 30 years old. In how many years' time will Shawn's age be 4 times David's age?

The teachers said that the 'Model' method—i.e. drawing of bar models, was only one of the heuristics students used when solving problems. At times when students tend to think that only bar

models must be drawn to solve problems they are constraining themselves. So, teachers do consciously engage students in a wide repertoire of problem types and also demonstrate the use of as many heuristics as possible when solving problems.

They felt that students in primary 4 would be able to find a solution to the problem. They may use systematic listing and look for ages that satisfy the condition 'Shawn's age is 4 times David's age" to find the answer (see left hand solution in Figure 14). However, if students choose to draw models they must be cognizant of the fact that the difference in the ages of Shawn and David will always be the same, which is 24 in this case. They may use a comparison model as shown in the right hand solution in Figure 16. All the teachers cautioned that many students may not encode the answer correctly and give the answer as 8 years. When they are guiding their students to solve the problem they would take the students through routines similar to those they articulated for problems 1 and 2.

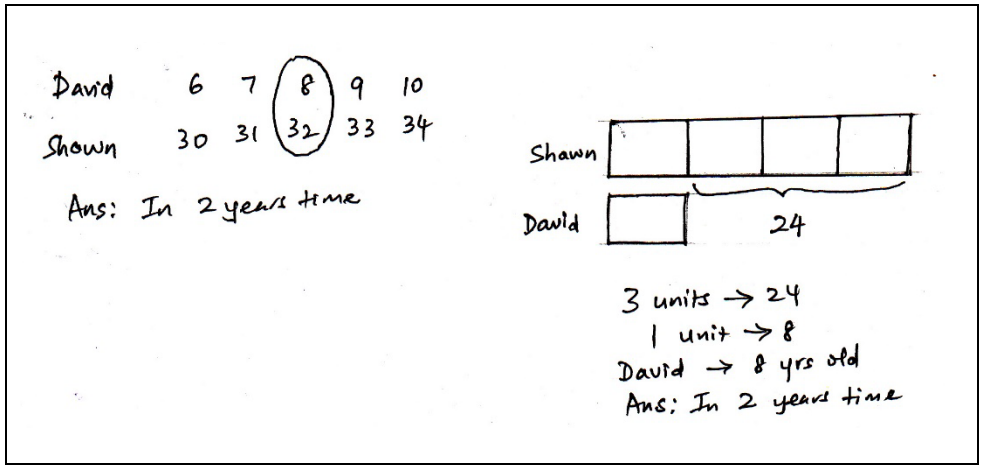


Fig. 14 Possible solutions for problem 3

Problem 4
Benedict used 84 sticks to make a total of 22 triangles and pentagons. A triangle was made using three sticks and a pentagon was made using 5 sticks. How many pentagons were there?

The teachers felt that this question definitely belonged to the schema of problems that use the guess and check heuristic for solving them. They were confident that their students in Primary 4 would be able to find the solution successfully but may not do so efficiently. In Primary 5 or 6, depending on the programme of the school, the heuristic *make a supposition* is introduced. They felt that students in these grades after been introduced to the heuristic could solve the problem more efficiently if they used it. Figure 15 shows a possible solution.

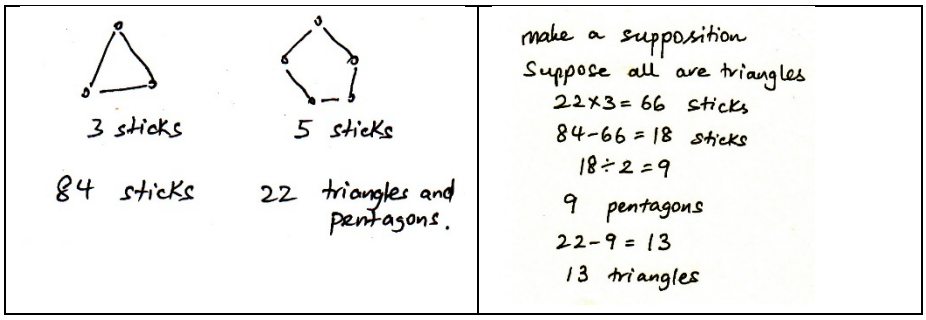


Fig. 15 Possible solution for problem 4

Without fail the teachers reiterated that they would engage their students in solving the problem by ensuring that they understood it, knew that to make triangles they need 3 sticks and 5 to make a pentagon. They also felt that as sticks are common manipulatives available in schools they would get students to work in small groups and explore the use of the heuristic guess and check to find the solution. They also wanted their students to experience that the guess and check process can be long and drawn out when the moves made are not systematic or guided by any mathematical observations. Then they may appreciate more efficient heuristics, when available to solve the problem. As for the other problems, they would ask students to share their approaches and if the students are in Primary 5 or 6 and have knowledge of the heuristic *make a supposition*, they would encourage students to examine both the approaches carefully and articulate their preferences.

Problem 5
In a basket, there are apples and oranges. 25% of the fruit are oranges. Mary buys more oranges and puts them into the basket to double their number. What percentage of the fruit are apples after Mary doubles the number of oranges?

All the teachers concurred that this problem could only be done by students in primary 6, as it requires knowledge of percentages and that 25% = ¼ . They also felt that this question was a rich one for classwork and had scope for teachers to engage students in making representations of their thoughts in several ways. When guiding their students in solving the problem they would

- Allow students time to read the problem / check for their understanding—ask students to explain what the givens and goal of the problem are.
- Engage students in acting out the problem / encourage students to discuss the problem in pairs / small groups.
- Check that students know that 25% = ¼ and what the fraction ¼ means? represents?
- Ask students to articulate what is given and what they have to find.
- Ask the students to draw a diagram to represent the problem.
- Ask a few students to share their diagrams on the board of the class.
- Scaffold students understanding of the before-after condition using real-life examples.
- Collect all the different representations on the board.
- Engage the whole class in examining the diagrams and guide them in making the links amongst them.
- Ask students to find "the percentage of apples after Mary doubles the number of oranges".
- Encourage them to share their answers and check for any errors they made.

Figure 16 shows possible representations. The teachers doubted if the students would be able to draw the bar models for the before and after states, right away without going through the process

of visualizing the groups of fruits in the basket (also representing 25% of fruit as 25 pieces of fruit) and the changes that take place after more oranges are added into the basket.

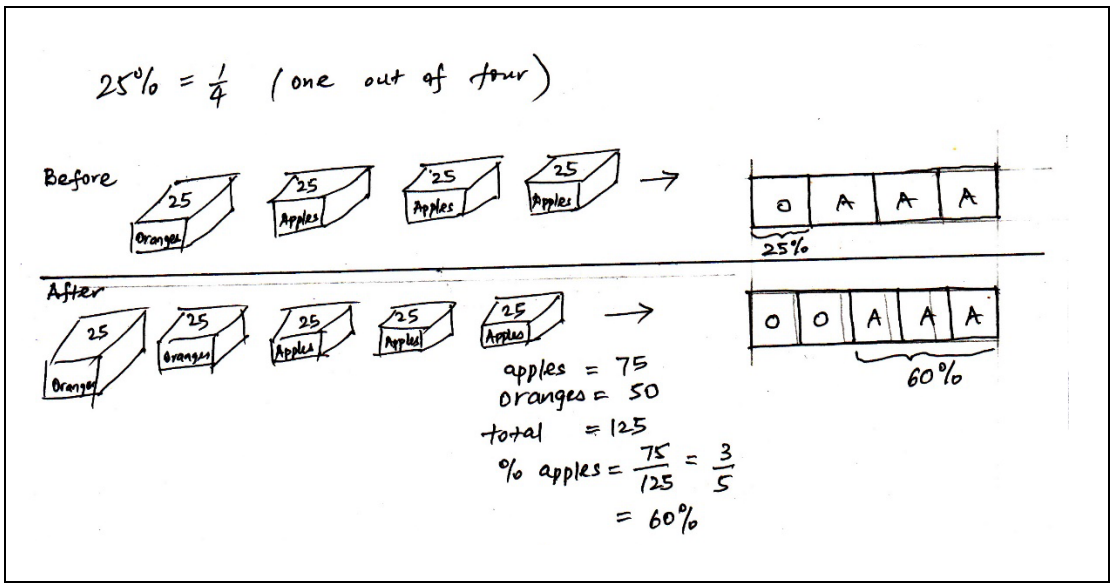


Fig. 16. Possible representations and solution for problem 5

4.3 Discussion

From the perceptions of the expert primary school mathematics teachers it is apparent that the 'Model' method is a tool students may use to solve whole number arithmetic word problems. It helps students who are having difficulty solving word problems, to represent and visualise relationships between the givens and goals of word problems. This result shows that the method has remained true to its original intent of a scaffold for students struggling with word problems irrespective of problem difficulty (MOE 2009). For topics such as fraction, ratio, and percentage the method help to concretise student thinking whilst making representations using the method. This perception echoes the findings of two local studies, namely those of Ng and Lee (2005) and Goh (2009).

The expert teachers also highlighted the process teachers would adopt when engaging students in solving word problems. The process is guided by Polya's (1957) stages of problem solving and Newman's (1983) strategies for error analysis. This finding illuminates the wider setting within which a heuristic facilitates the solution of a word problem. It is also apparent from the teachers' perceptions that choice of heuristic, be it making representations using the 'Model' method or any other, depends on students' knowledge and preference. This finding is critical as it dispels the myth that the Singapore primary school mathematics curriculum often referred to as "Singapore Math" *is* the 'Model' method.

The limitations of the small scale study are twofold. Firstly, though the set of five problems were diverse and representative of the curriculum they may have skewed the responses of the participants. Secondly, as the study used a purposive sample it is unable to shed light on the

perceptions of novice teachers, who are often developing their competencies under the guidance of expert teachers, such as the ones who participated in the study.

4. Conclusion

This paper has attempted to present a succinct consolidation of the why, what and how of the 'Model' method. The history of the method in the Singapore primary school mathematics curriculum has shown us that the method has rightly addressed student difficulty with word problems whilst being adopted in the instructional materials as a tool for representing and visualising relationships when solving word problems. It has also enhanced making sense of fractions, ratios and percentages through representations, such as bar models. The few studies, carried out in Singapore and elsewhere, all affirm the efficacy of the method. Lastly expert teachers of primary mathematics shed light on the wider setting of solving word problems of which the 'Model' method is a part of and not the part of in Singapore schools.

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