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Proving a conjecture on chromatic polynomials by counting the number of acyclic orientations

Fengming Dong*, Jun Ge, Helin Gong

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Abstract

The chromatic polynomial $P(G, x)$ of a graph G of order n can be expressed as $\sum_{i=1}^n (-1)^{n-i} a_i x^i$, where a_i is interpreted as the number of broken-cycle free spanning subgraphs of G with exactly i components. The parameter $\epsilon(G) = \frac{\sum_{i=1}^n (n-i)a_i}{\sum_{i=1}^n a_i}$ is the mean size of a broken-cycle-free spanning subgraph of G . In this article, we confirm and strengthen a conjecture proposed by Lundow and Markström in 2006 that $\epsilon(T_n) < \epsilon(G) < \epsilon(K_n)$ holds for any connected graph G of order n which is neither the complete graph K_n nor a tree T_n of order n . The most crucial step of our proof is to obtain the interpretation of all a_i 's by the number of acyclic orientations of G .

Keywords: chromatic polynomial; graph; acyclic orientation; combinatorial interpretation

Mathematics Subject Classification (2010): 05C31, 05C20

1 Introduction

All graphs considered in this paper are simple graphs. For any graph $G = (V, E)$ and any positive integer k , a *proper k -coloring* f of G is a mapping $f : V \rightarrow \{1, 2, \dots, k\}$ such that $f(u) \neq f(v)$ holds whenever $uv \in E$. The chromatic polynomial of G is the function $P(G, x)$ such that $P(G, k)$ counts the number of proper k -colorings of G for any positive integer k . In this article, the variable x in $P(G, x)$ is a real number.

The study of chromatic polynomials is one of the most active areas in graph theory. For basic concepts and properties on chromatic polynomials, we refer the reader to the

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monograph [5]. For the most celebrated results on this topic, we recommend surveys [4, 10, 14, 15].

The first interpretation of the coefficients of $P(G, x)$ was provided by Whitney [23]: for any simple graph G of order n and size m ,

$$P(G, x) = \sum_{i=1}^n \left(\sum_{r=0}^m (-1)^r N(i, r) \right) x^i, \quad (1)$$

where $N(i, r)$ is the number of spanning subgraphs of G with exactly i components and r edges. Whitney further simplified (1) by introducing the notion of broken cycles. Let $\eta : E \rightarrow \{1, 2, \dots, |E|\}$ be a bijection. For any cycle C in G , the path $C - e$ is called a *broken cycle* of G with respect to η , where e is the edge on C with $\eta(e) \leq \eta(e')$ for every edge e' on C . When there is no confusion, a broken cycle of G is always assumed to be with respect to a bijection $\eta : E \rightarrow \{1, 2, \dots, |E|\}$.

Theorem 1 ([23]). *Let $G = (V, E)$ be a graph of order n and $\eta : E \rightarrow \{1, 2, \dots, |E|\}$ be a bijection. Then,*

$$P(G, x) = \sum_{i=1}^n (-1)^{n-i} a_i(G) x^i, \quad (2)$$

where $a_i(G)$ is the number of spanning subgraphs of G with $n - i$ edges and i components which do not contain broken cycles.

Let G be a simple graph of order n . When there is no confusion, $a_i(G)$ is written as a_i for short. Clearly, by Theorem 1, $P(G, x)$ is indeed a polynomial in x in which the constant term is 0, the leading coefficient a_n is 1 and all coefficients are integers alternating in signs. Thus, $(-1)^n P(G, x) > 0$ holds for all $x < 0$.

The concept of broken cycles has the following connection with Tutte's work of expressing the Tutte polynomial $\mathbf{T}_G(x, y)$ of a connected graph G in terms of spanning trees [2, 22]:

$$\mathbf{T}_G(x, y) = \sum_T x^{ia_\omega(T)} y^{ea_\omega(T)}, \quad (3)$$

where the sum runs over all spanning trees of G and $ia_\omega(T)$ and $ea_\omega(T)$ are respectively the internal and external activities of T with respect to a bijection $\omega : E \rightarrow \{1, 2, \dots, |E|\}$. If we take ω to be η , then $ea_\eta(T)$ is exactly the number of edges $e \in E(G) \setminus E(T)$ such that $\eta(e) \leq \eta(e')$ holds for all edges e' on the unique cycle C of $T \cup e$. As G is a simple graph, $ea_\eta(T)$ equals the number of broken cycles contained in T with respect to η . In

particular, $ea_\eta(T) = 0$ if and only if T does not contain broken cycles with respect to η . By Theorem 1, $a_1(G)$ is the number of spanning trees T of G with $ea_\eta(T) = 0$. If

$$\mathbf{T}_G(x, y) = \sum_{i \geq 0, j \geq 0} c_{i,j} x^i y^j, \quad (4)$$

then $a_1(G) = \sum_{i \geq 0} c_{i,0} = \mathbf{T}_G(1, 0)$.

As in [11], for $i = 0, 1, 2, \dots, n-1$, we define $b_i(G)$ (or simply b_i) as the probability that a randomly chosen broken-cycle-free spanning subgraph of G has size i . Then

$$b_i = \frac{a_{n-i}}{a_1 + a_2 + \dots + a_n}, \quad \forall i = 0, 1, \dots, n-1. \quad (5)$$

Let $\epsilon(G)$ denote the mean size of a broken-cycle-free spanning subgraph of G . Then

$$\epsilon(G) = \sum_{i=0}^{n-1} i b_i = \frac{(n-1)a_1 + (n-2)a_2 + \dots + a_{n-1}}{a_1 + a_2 + \dots + a_n}. \quad (6)$$

An elementary property of $\epsilon(G)$ is given below.

Proposition 1 ([11]). For any graph G of order n , $\epsilon(G) = n + \frac{P'(G, -1)}{P(G, -1)}$.

Let T_n denote a tree of order n and K_n denote the complete graph of order n . By Proposition 1, $\epsilon(T_n) = \frac{n-1}{2}$, while

$$\epsilon(K_n) = n - \left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) \sim n - \log n - \gamma \quad (7)$$

as $n \rightarrow \infty$, where $\gamma \approx 0.577216$ is the Euler-Mascheroni constant.

Lundow and Markström [11] proposed the following conjecture on $\epsilon(G)$.

Conjecture 1 ([11]). For any connected graph G of order n , where $n \geq 4$, if G is neither K_n nor a T_n , then $\epsilon(T_n) < \epsilon(G) < \epsilon(K_n)$.

In this paper, we aim to prove and strengthen Conjecture 1. For any graph G , define the function $\epsilon(G, x)$ as follows:

$$\epsilon(G, x) = \frac{P'(G, x)}{P(G, x)}. \quad (8)$$

By Proposition 1, $\epsilon(G) = n + \epsilon(G, -1)$ holds for every graph G of order n . Thus, for any graphs G and H of the same order, $\epsilon(G) < \epsilon(H)$ if and only if $\epsilon(G, -1) < \epsilon(H, -1)$. Conjecture 1 is equivalent to the statement that $\epsilon(T_n, -1) < \epsilon(G, -1) < \epsilon(K_n, -1)$ holds for any connected graph G of order n which is neither K_n nor a T_n .

A graph Q is said to be *chordal* if $Q[V(C)] \not\cong C$ for every cycle C of Q with $|V(C)| \geq 4$, where $Q[V']$ is the subgraph of Q induced by V' for $V' \subseteq V(G)$. In Section 2, we will establish the following result.

Theorem 2. For any graph G , if Q is a chordal and proper spanning subgraph of G , then $\epsilon(G, x) > \epsilon(Q, x)$ holds for all $x < 0$.

Note that any tree is a chordal graph and any connected graph contains a spanning tree. Thus, we have the following corollary which obviously implies the first part of Conjecture 1.

Corollary 1. For any connected graph G of order n which is not a tree, $\epsilon(G, x) > \epsilon(T_n, x)$ holds for all $x < 0$.

The second part of Conjecture 1 is extended to the inequality $\epsilon(K_n, x) > \epsilon(G, x)$ for any non-complete graph G of order n and all $x < 0$. In order to prove this inequality, we will show in Section 3 that it suffices to establish the following result.

Theorem 3. For any non-complete graph $G = (V, E)$ of order n ,

$$(-1)^n(x - n + 1) \sum_{u \in V} P(G - u, x) + (-1)^{n+1} n P(G, x) > 0 \quad (9)$$

holds for all $x < 0$.

Note that the left-hand side of (9) vanishes when $G \cong K_n$. Theorem 3 will be proved in Section 5, based on Greene & Zaslavsky's interpretation in [8] for coefficients $a_i(G)$'s of $P(G, x)$ by acyclic orientations introduced in Section 4. By applying Theorem 3 and two lemmas in Section 3, we will finally prove the second main result in this article.

Theorem 4. For any non-complete graph G of order n , $\epsilon(G, x) < \epsilon(K_n, x)$ holds for all $x < 0$.

2 Proof of Theorem 2

A vertex u in a graph G is called a *simplicial vertex* if $\{u\} \cup N_G(u)$ is a clique of G , where $N_G(u)$ is the set of vertices in G which are adjacent to u . For a simplicial vertex u of G , $P(G, x)$ has the following property (see [5, 13, 14]):

$$P(G, x) = (x - d(u))P(G - u, x), \quad (10)$$

where $G - u$ is the subgraph of G induced by $V - \{u\}$ and $d(u)$ is the degree of u in G . By (10), it is not difficult to show the following.

Proposition 2. If u is a simplicial vertex of a graph G , then

$$\epsilon(G, x) = \frac{1}{x - d(u)} + \epsilon(G - u, x). \quad (11)$$

It has been shown that a graph Q of order n is chordal if and only if Q has an ordering u_1, u_2, \dots, u_n of its vertices such that u_i is a simplicial vertex in $Q[\{u_1, u_2, \dots, u_i\}]$ for all $i = 1, 2, \dots, n$ (see [3, 6]). Such an ordering of vertices in Q is called a *perfect elimination ordering* of Q . For any perfect elimination ordering u_1, u_2, \dots, u_n of a chordal graph Q , by Proposition 2,

$$\epsilon(Q, x) = \sum_{i=1}^n \frac{1}{x - d_{Q_i}(u_i)}, \quad (12)$$

where Q_i is the subgraph $Q[\{u_1, u_2, \dots, u_i\}]$.

Now we are ready to prove Theorem 2.

Proof of Theorem 2: Let G be any graph of order n and Q be any chordal and proper spanning subgraph of G . When $n \leq 3$, it is not difficult to verify that $\epsilon(G, x) > \epsilon(Q, x)$ holds for all $x < 0$.

Suppose that Theorem 2 fails and $G = (V, E)$ is a counter-example to this result such that $|V| + |E|$ has the minimum value among all counter-examples. Thus the result holds for any graph H with $|V(H)| + |E(H)| < |V| + |E|$ and any chordal and proper spanning subgraph Q' of H , but G has a chordal and proper spanning subgraph Q such that $\epsilon(G, x) \leq \epsilon(Q, x)$ holds for some $x < 0$.

We will establish the following claims. Let u_1, u_2, \dots, u_n be a perfect elimination ordering of Q and $Q_i = Q[\{u_1, \dots, u_i\}]$ for all $i = 1, 2, \dots, n$. So u_i is a simplicial vertex of Q_i for all $i = 1, 2, \dots, n$.

Claim 1: u_n is not a simplicial vertex of G .

Note that $Q - u_n$ is chordal and a spanning subgraph of $G - u_n$. By the assumption on the minimality of $|V| + |E|$, $\epsilon(G - u_n, x) \geq \epsilon(Q - u_n, x)$ holds for all $x < 0$, where the inequality is strict whenever $Q - u_n \not\cong G - u_n$.

Clearly $d_G(u_n) \geq d_Q(u_n)$. As Q is a proper subgraph of G , $d_G(u_n) > d_Q(u_n)$ in the case that $G - u_n \cong Q - u_n$. If u_n is also a simplicial vertex of G , then by Proposition 2,

$$\epsilon(G, x) = \frac{1}{x - d_G(u_n)} + \epsilon(G - u_n, x), \quad \epsilon(Q, x) = \frac{1}{x - d_Q(u_n)} + \epsilon(Q - u_n, x), \quad (13)$$

implying that $\epsilon(G, x) > \epsilon(Q, x)$ holds for all $x < 0$, a contradiction. Hence Claim 1 holds.

Claim 2: $d_G(u_n) > d_Q(u_n)$.

Clearly $d_G(u_n) \geq d_Q(u_n)$. Since u_n is a simplicial vertex of Q and Q is a subgraph of G , $d_G(u_n) = d_Q(u_n)$ implies that u_n is a simplicial vertex of G , contradicting Claim 1. Thus Claim 2 holds.

For any edge e in G , let $G - e$ be the graph obtained from G by deleting e . Let G/e be the graph obtained from G by contracting e and replacing multiple edges, if any arise, by single edges.

Claim 3: For any $e = u_nv \in E - E(Q)$, both $\epsilon(G - e, x) \geq \epsilon(Q, x)$ and $\epsilon(G/e, x) \geq \epsilon(Q - u_n, x)$ hold for all $x < 0$.

As $e = u_nv \in E - E(Q)$, Q is a spanning subgraph of $G - e$ and $Q - u_n$ is a spanning subgraph of G/e . As both Q and $Q - u_n$ are chordal, by the assumption on the minimality of $|V| + |E|$, the theorem holds for both $G - e$ and G/e . Thus this claim holds.

Claim 4: $\epsilon(G, x) > \epsilon(Q, x)$ holds for all $x < 0$.

By Claim 2, there exists $e = u_nv \in E - E(Q)$. By Claim 3, $\epsilon(G - e, x) \geq \epsilon(Q, x)$ and $\epsilon(G/e, x) \geq \epsilon(Q - u_n, x)$ hold for all $x < 0$. By (8) and (12),

$$\begin{aligned} & (\epsilon(G - e, x) - \epsilon(Q, x)) \times (-1)^n P(G - e, x) \\ &= (-1)^n P'(G - e, x) + (-1)^{n+1} P(G - e, x) \sum_{i=1}^n \frac{1}{x - d_{Q_i}(u_i)}. \end{aligned} \quad (14)$$

As $(-1)^n P(G - e, x) > 0$ and $\epsilon(G - e, x) \geq \epsilon(Q, x)$ for all $x < 0$, the left-hand side of (14) is non-negative for $x < 0$, implying that the right-hand side of (14) is also non-negative for $x < 0$, i.e.,

$$(-1)^n P'(G - e, x) + (-1)^{n+1} P(G - e, x) \sum_{i=1}^n \frac{1}{x - d_{Q_i}(u_i)} \geq 0, \quad \forall x < 0. \quad (15)$$

As u_1, \dots, u_{n-1} is a perfect elimination ordering of $Q - u_n$ and $\epsilon(G/e, x) \geq \epsilon(Q - u_n, x)$ holds for all $x < 0$, similarly we have:

$$(-1)^{n-1} P'(G/e, x) + (-1)^n P(G/e, x) \sum_{i=1}^{n-1} \frac{1}{x - d_{Q_i}(u_i)} \geq 0, \quad \forall x < 0. \quad (16)$$

As $(-1)^{n-1} P(G/e, x) > 0$ holds for all $x < 0$, (16) implies that

$$\begin{aligned} & (-1)^{n-1} P'(G/e, x) + (-1)^n P(G/e, x) \sum_{i=1}^n \frac{1}{x - d_{Q_i}(u_i)} \\ & \geq \frac{(-1)^n P(G/e, x)}{x - d_{Q_n}(u_n)} > 0, \quad \forall x < 0. \end{aligned} \quad (17)$$

By the deletion-contraction formula for chromatic polynomials,

$$P(G, x) = P(G - e, x) - P(G/e, x), \quad P'(G, x) = P'(G - e, x) - P'(G/e, x). \quad (18)$$

Then (15), (17) and (18) imply that

$$(-1)^n P'(G, x) + (-1)^{n+1} P(G, x) \sum_{i=1}^n \frac{1}{x - d_{Q_i}(u_i)} > 0, \quad \forall x < 0. \quad (19)$$

By (8) and (12), inequality (19) implies that

$$(\epsilon(G, x) - \epsilon(Q, x))(-1)^n P(G, x) > 0, \quad \forall x < 0. \quad (20)$$

Since $(-1)^n P(G, x) > 0$ holds for all $x < 0$, inequality (20) implies Claim 4.

As Claim 4 contradicts the assumption of G , there are no counter-examples to this result and the theorem is proved. \square

3 An approach for proving Theorem 4

In this section, we will mainly show that, in order to prove Theorem 4, it suffices to prove Theorem 3. By (12), we have

$$\epsilon(K_n, x) = \sum_{i=0}^{n-1} \frac{1}{x-i}. \quad (21)$$

Thus,

$$\epsilon(K_n, x) - \epsilon(G, x) = \frac{(-1)^n}{P(G, x)} \left((-1)^n P(G, x) \sum_{i=0}^{n-1} \frac{1}{x-i} + (-1)^{n+1} P'(G, x) \right). \quad (22)$$

For any graph G of order n , define

$$\xi(G, x) = (-1)^n P(G, x) \sum_{i=0}^{n-1} \frac{1}{x-i} + (-1)^{n+1} P'(G, x). \quad (23)$$

Note that $\xi(G, x) \equiv 0$ if G is a complete graph. For any non-complete graph G and any $x < 0$, we have $(-1)^n P(G, x) > 0$ and so (22) implies that $\epsilon(K_n, x) - \epsilon(G, x) > 0$ if and only if $\xi(G, x) > 0$.

Proposition 3. *Theorem 4 holds if and only if $\xi(G, x) > 0$ holds for every non-complete graph G and all $x < 0$.*

It can be easily verified that $\xi(G, x) > 0$ holds for all non-complete graphs G of order at most 3 and all $x < 0$. For the general case, we will prove it by induction. In the rest of this section, we will find a relation between $\xi(G, x)$ and $\xi(G - u, x)$ for a vertex u in G in two cases. Lemma 1 is for the case when u is a simplicial vertex and Lemma 3 when $d(u) \geq 1$. We then explain why Theorem 3 implies $\xi(G, x) > 0$ for all non-complete graphs G and all $x < 0$.

Lemma 1. *Let G be a graph of order n . If u is a simplicial vertex of G with $d(u) = d$, then*

$$\xi(G, x) = (d-x)\xi(G-u, x) + \frac{(-1)^{n-1}(n-1-d)P(G-u, x)}{n-1-x}. \quad (24)$$

Proof. As u is a simplicial vertex of G with $d(u) = d$, $P(G, x) = (x - d)P(G - u, x)$ by (10). Thus $P'(G, x) = P(G - u, x) + (x - d)P'(G - u, x)$. By (23),

$$\begin{aligned} \xi(G, x) &= (-1)^n(x - d)P(G - u, x) \sum_{i=0}^{n-1} \frac{1}{x - i} + (-1)^{n+1}(P(G - u, x) + (x - d)P'(G - u, x)) \\ &= (d - x)\xi(G - u, x) + \frac{(-1)^n(x - d)P(G - u, x)}{x - n + 1} + (-1)^{n+1}P(G - u, x) \\ &= (d - x)\xi(G - u, x) + \frac{(-1)^{n-1}(n - 1 - d)P(G - u, x)}{n - 1 - x}. \end{aligned} \quad (25)$$

□

Note that $d \leq n - 1$ and $(-1)^{n-1}P(G - u, x) > 0$ holds for all $x < 0$, implying that the second term in the right-hand side of (24) is non-negative. Thus, if u is a simplicial vertex of G and $x < 0$, by Lemma 1, $\xi(G - u, x) > 0$ implies that $\xi(G, x) > 0$.

Now consider the case that u is a vertex in G with $d(u) = d \geq 1$. Assume that $N(u) = \{u_1, u_2, \dots, u_d\}$. For any $i = 1, 2, \dots, d - 1$, let G_i denote the graph obtained from $G - u$ by adding edges joining u_i to u_j whenever $u_i u_j \notin E(G)$ for all j with $i + 1 \leq j \leq d$. Thus, u_i is adjacent to u_j in G_i for all j with $i + 1 \leq j \leq d$. In the case that u is a simplicial vertex of G , $G_i \cong G - u$ for all $i = 1, 2, \dots, d - 1$. By applying the deletion-contraction formula for chromatic polynomials (see [5, 13]), $P(G, x)$ can be expressed in terms of $P(G - u, x)$ and $P(G_i, x)$ for $i = 1, 2, \dots, d - 1$.

Lemma 2. *Let u be a vertex in G with $d(u) = d \geq 1$ and for $i = 1, 2, \dots, d - 1$, let G_i be the graph defined above. Then,*

$$P(G, x) = (x - 1)P(G - u, x) - \sum_{i=1}^{d-1} P(G_i, x). \quad (26)$$

Proof. For $1 \leq i \leq d$, let E_i denote the set of edges uu_j in G for $j = 1, 2, \dots, i - 1$. So $|E_i| = i - 1$ and $E_1 = \emptyset$. For any i with $1 \leq i \leq d - 1$, applying the deletion-contraction formula for chromatic polynomials to edge uu_i in $G - E_i$, the graph obtained from G by removing all edges in E_i , we have

$$P(G - E_i, x) = P(G - E_{i+1}, x) - P((G - E_i)/uu_i, x) = P(G - E_{i+1}, x) - P(G_i, x), \quad (27)$$

where the last equality follows from the fact that $(G - E_i)/uu_i \cong G_i$ by the assumption of G_i . Thus, by (27),

$$P(G, x) = P(G - E_1, x) = P(G - E_d, x) - \sum_{i=1}^{d-1} P(G_i, x). \quad (28)$$

As u is of degree 1 in $G - E_d$, $P(G - E_d, x) = (x - 1)P(G - u, x)$. Hence (26) follows. □

Lemma 3. Let G be a graph of order n and let u be a vertex of G with $d(u) = d \geq 1$.

Then

$$\xi(G, x) = (1-x)\xi(G-u, x) + \sum_{i=1}^{d-1} \xi(G_i, x) + \frac{(-1)^n [(x-n+1)P(G-u, x) - P(G, x)]}{n-x-1}, \quad (29)$$

where G_1, \dots, G_{d-1} are graphs defined above.

Proof. By (26), we have

$$P'(G, x) = P(G-u, x) + (x-1)P'(G-u, x) - \sum_{i=1}^{d-1} P'(G_i, x). \quad (30)$$

Thus

$$\begin{aligned} \xi(G, x) &= (-1)^n P(G, x) \sum_{j=0}^{n-1} \frac{1}{x-j} + (-1)^{n+1} P'(G, x) \\ &= (-1)^n \left[(x-1)P(G-u, x) - \sum_{i=1}^{d-1} P(G_i, x) \right] \sum_{j=0}^{n-1} \frac{1}{x-j} \\ &\quad + (-1)^{n+1} \left[P(G-u, x) + (x-1)P'(G-u, x) - \sum_{i=1}^{d-1} P'(G_i, x) \right] \\ &= (1-x) \left[(-1)^{n-1} P(G-u, x) \sum_{j=0}^{n-2} \frac{1}{x-j} + (-1)^n P'(G-u, x) \right] \\ &\quad + \sum_{i=1}^{d-1} \left[(-1)^{n-1} P(G_i, x) \sum_{j=0}^{n-2} \frac{1}{x-j} + (-1)^n P'(G_i, x) \right] + (-1)^{n+1} P(G-u, x) \\ &\quad + (-1)^n \left[\frac{(x-1)P(G-u, x)}{x-(n-1)} - \frac{1}{x-(n-1)} \sum_{i=1}^{d-1} P(G_i, x) \right] \\ &= (1-x)\xi(G-u, x) + \sum_{i=1}^{d-1} \xi(G_i, x) \\ &\quad + \frac{(-1)^n [(x-n+1)P(G-u, x) - P(G, x)]}{n-x-1}, \end{aligned} \quad (31)$$

where the last expression follows from (26) and the definitions of $\xi(G-u, x)$ and $\xi(G_i, x)$.

The result then follows. \square

It is known that $\xi(G, x) > 0$ holds for all non-complete graphs G of order at most 3 and all $x < 0$. For any non-complete graph G of order $n \geq 4$, by Lemma 1, $\xi(G-u, x) > 0$ implies $\xi(G, x) > 0$ for each simplicial vertex u in G and all $x < 0$; by Lemma 3, for any

$x < 0$, $\xi(G - u, x) > 0$ implies $\xi(G, x) > 0$ whenever u is an non-isolated vertex in G satisfying the following inequality:

$$(-1)^n((x - n + 1)P(G - u, x) - P(G, x)) > 0. \quad (32)$$

Note that the left-hand side of (32) vanishes when G is K_n . Also notice that there exist non-complete graph G and some vertex u in G such that inequality (32) does not hold for some $x < 0$. For example, if G is the complete bipartite graph $K_{2,3}$ and u is a vertex of degree 3 in G , then (32) fails for all real x with $-2.3 < x < 0$. However, to prove that for any $x < 0$, there exists some vertex u in G such that inequality (32) holds, it suffices to prove the following inequality (i.e., Theorem 3):

$$(-1)^n(x - n + 1) \sum_{u \in V} P(G - u, x) + (-1)^{n+1}nP(G, x) > 0 \quad (33)$$

for any non-complete graph $G = (V, E)$ of order n and all $x < 0$.

By Proposition 3 and inequality (32), to prove Theorem 4, we can now just focus on proving inequality (33) (i.e., Theorem 3). The proof of Theorem 3 will be given in Section 5 based on the interpretations for the coefficients of chromatic polynomials introduced in Section 4.

4 Combinatorial interpretations for coefficients of $P(G, x)$

Let $G = (V, E)$ be any graph. In this section, we will introduce Greene & Zaslavsky's combinatorial interpretation in [8] for the coefficients of $P(G, x)$ in terms of acyclic orientations. The result will be applied in the next section to prove Theorem 3.

An orientation D of G is called *acyclic* if D does not contain any directed cycle. Let $\alpha(G)$ be the number of acyclic orientations of a graph G . In [18], Stanley gave a nice combinatorial interpretation of $(-1)^nP(G, -k)$ for any positive integer k in terms of acyclic orientations of G . In particular, he proved:

Theorem 5 ([18]). *For any graph G of order n , $(-1)^nP(G, -1) = \alpha(G)$, i.e.,*

$$\sum_{i=1}^n a_i(G) = \alpha(G). \quad (34)$$

In a digraph D , any vertex of D with in-degree (resp. out-degree) zero is called a *source* (resp. *sink*) of D . It is well known that any acyclic digraph has at least one source and at least one sink. If v is an isolated vertex of G , then v is a source and also a sink in any orientation of G .

For any $v \in V$, let $\alpha(G, v)$ be the number of acyclic orientations of G with v as its unique source. Clearly $\alpha(G, v) = 0$ if and only if G is not connected. In 1983, Greene and Zaslavsky [8] showed that $a_1(G) = \alpha(G, v)$.

Theorem 6 ([8]). *For any graph $G = (V, E)$, $a_1(G) = \alpha(G, v)$ holds for every $v \in V$.*

This theorem was proved originally by using the theory of hyperplane arrangements. See [7] for three other nice proofs.

By Whitney's Broken-cycle Theorem (i.e., Theorem 1), $a_i(G)$ equals the number of spanning subgraphs of G with i components and $n - i$ edges, containing no broken cycles of G . In particular, $a_1(G)$ is the number of spanning trees of G containing no broken cycles of G . Now we have two different combinatorial interpretations for a_1 . For any $a_i(G)$, $2 \leq i \leq n$, its combinatorial interpretation can be obtained by applying these two different combinatorial interpretations for a_1 .

Let $\mathcal{P}_i(V)$ be the set of partitions $\{V_1, V_2, \dots, V_i\}$ of V such that $G[V_j]$ is connected for all $j = 1, 2, \dots, i$ and let $\beta_i(G)$ be the number of ordered pairs (P_i, F) , where

- (a) $P_i = \{V_1, V_2, \dots, V_i\} \in \mathcal{P}_i(V)$;
- (b) F is a spanning forest of G with exactly i components T_1, T_2, \dots, T_i , where each T_j is a spanning tree of $G[V_j]$ containing no broken cycles of G .

For any subgraph H of G , let $\tilde{\tau}(H)$ be the number of spanning trees of H containing no broken cycles of G . By Theorem 1, $\tilde{\tau}(H) = a_1(H)$ holds and the next result follows.

Theorem 7. *For any graph G and any $1 \leq i \leq n$,*

$$a_i(G) = \beta_i(G) = \sum_{\{V_1, \dots, V_i\} \in \mathcal{P}_i(V)} \prod_{j=1}^i \tilde{\tau}(G[V_j]). \quad (35)$$

Now let $V = \{1, 2, \dots, n\}$. For any $i : 1 \leq i \leq n$ and any vertex $v \in V$, let $\mathcal{OP}_{i,v}(V)$ be the family of ordered partitions (V_1, V_2, \dots, V_i) of V such that

- (a) $\{V_1, V_2, \dots, V_i\} \in \mathcal{P}_i(V)$, where $v \in V_1$;
- (b) for $j = 2, \dots, i$, the minimum number in the set $\bigcup_{j \leq s \leq i} V_s$ is within V_j .

Clearly, for any $v \in V$ and any $\{V_1, V_2, \dots, V_i\} \in \mathcal{P}_i(V)$, there is exactly one permutation $(\pi_1, \pi_2, \dots, \pi_i)$ of $1, 2, \dots, i$ such that $(V_{\pi_1}, V_{\pi_2}, \dots, V_{\pi_i}) \in \mathcal{OP}_{i,v}(V)$.

By Theorem 6, $\tilde{\tau}(G[V_j]) = \alpha(G[V_j], u)$ holds for any vertex u in $G[V_j]$ and Theorem 7 is equivalent to a result in [8] which we illustrate differently below.

Theorem 8 ([8], Theorem 7.4). *For any $v \in V$ and any $1 \leq i \leq n$,*

$$a_i(G) = \sum_{(V_1, \dots, V_i) \in \mathcal{OP}_{i,v}(V)} \alpha(G[V_1], v) \prod_{j=2}^i \alpha(G[V_j], m_j), \quad (36)$$

where m_j is the minimum number in V_j for $j = 2, \dots, i$.

Note that the theorem above indicates that the right hand side of (36) is independent of the choice of v . Thus, for any $1 \leq i \leq n$,

$$na_i(G) = \sum_{v \in V} \sum_{(V_1, \dots, V_i) \in \mathcal{OP}_{i,v}(V)} \alpha(G[V_1], v) \prod_{j=2}^i \alpha(G[V_j], m_j). \quad (37)$$

Let $P^{(i)}(G, x)$ be the i -th derivative of $P(G, x)$. Very recently, Bernardi and Nadeau [1] gave an interpretation of $P^{(i)}(G, -j)$ for any nonnegative integers i and j in terms of acyclic orientations. When $i = 0$, their result is exactly Theorem 5 due to Stanley [18]; and when $j = 0$, it is Theorem 8 due to Greene & Zaslavsky [8].

5 Proofs of Theorems 3 and 4

By the explanation in Section 3, to prove Theorem 4, it suffices to prove Theorem 3. In this section, we will prove Theorem 3 by showing that the coefficient of x^i in the expansion of the left-hand side of (9) in Theorem 3 is of the form $(-1)^i d_i$ with $d_i \geq 0$ for all $i = 1, 2, \dots, n$. Furthermore, $d_i > 0$ holds for some i when G is not complete.

We first establish the following result.

Lemma 4. *Let $G = (V, E)$ be a non-complete graph of order $n \geq 3$ and component number c .*

- (i). *If $c = 1$ and G is not the n -cycle C_n , then there exist non-adjacent vertices u_1, u_2 of G such that $G - \{u_1, u_2\}$ is connected.*
- (ii). *If $2 \leq c \leq n - 1$, then for any integer i with $c \leq i \leq n - 1$, there exists a partition V_1, V_2, \dots, V_i of V such that $G[V_j]$ is connected for all $j = 2, \dots, i$ and $G[V_1]$ has exactly two components one of which is an isolated vertex.*

Proof. (i). As $c = 1$, G is connected. As G is non-complete, the result is trivial when G is 3-connected.

If G is not 2-connected, choose vertices u_1 and u_2 from distinct blocks B_1 and B_2 of G such that both u_1 and u_2 are not cut-vertices of G . Then $u_1 u_2 \notin E(G)$ and $G - \{u_1, u_2\}$ is connected.

Now consider the case that G is 2-connected but not 3-connected. Since G is not C_n , there exists a vertex w such that $d(w) \geq 3$. If $d(w) = n - 1$, then $G - \{u_1, u_2\}$ is connected for any two non-adjacent vertices u_1 and u_2 in G . If $G - w$ is 2-connected and $d(w) \leq n - 2$, then $G - \{w, u\}$ is connected for any $u \in V - N_G(w)$. If $G - w$ is not 2-connected, then $G - w$ contains two non-adjacent vertices u_1, u_2 such that $G - \{w, u_1, u_2\}$ is connected, implying that $G - \{u_1, u_2\}$ is connected as $d(w) \geq 3$.

(ii). Let G_1, G_2, \dots, G_c be the components of G with $|V(G_1)| \geq |V(G_j)|$ for all $j = 1, 2, \dots, c$. As $c \leq n - 1$, $|V(G_1)| \geq 2$. Choose $u \in V(G_1)$ such that $G_1 - u$ is connected. Then $V(G_2) \cup \{u\}, V(G_1) - \{u\}, V(G_3), \dots, V(G_c)$ is a partition of V satisfying the condition in (ii) for $i = c$.

Assume that (ii) holds for $i = k$, where $c \leq k < n - 1$, and V_1, V_2, \dots, V_k is a partition of V satisfying the condition in (ii). Then $G[V_1]$ has an isolated vertex u and $G[V'_1]$ is connected, where $V'_1 = V_1 - \{u\}$. Since $k \leq n - 2$, either $|V'_1| \geq 2$ or $|V_j| \geq 2$ for some $j \geq 2$.

If $|V'_1| \geq 2$, then V'_1 has a partition $V'_{1,1}, V'_{1,2}$ such that both $G[V'_{1,1}]$ and $G[V'_{1,2}]$ are connected, implying that $V'_{1,1} \cup \{u\}, V'_{1,2}, V_2, V_3, \dots, V_k$ is a partition of V satisfying the condition in (ii) for $i = k + 1$.

Similarly, if $|V_j| \geq 2$ for some $j \geq 2$ (say $j = 2$), then V_2 has a partition $V_{2,1}, V_{2,2}$ such that both $G[V_{2,1}]$ and $G[V_{2,2}]$ are connected, implying that $V_1, V_{2,1}, V_{2,2}, V_3, \dots, V_k$ is a partition of V satisfying the condition in (ii) for $i = k + 1$. \square

For any graph $G = (V, E)$ of order n , write

$$(-1)^n \left[(x - n + 1) \sum_{u \in V(G)} P(G - u, x) - nP(G, x) \right] = \sum_{i=1}^n (-1)^i d_i x^i. \quad (38)$$

By comparing coefficients, it can be shown that

$$d_i = \sum_{u \in V(G)} [a_{i-1}(G - u) + (n - 1)a_i(G - u)] - na_i(G), \quad \forall i = 1, 2, \dots, n. \quad (39)$$

It is obvious that when G is the complete graph K_n , the left-hand side of (38) vanishes and thus $d_i = 0$ for all $i = 1, 2, \dots, n$. Now we consider the case that G is not complete.

Proposition 4. *Let $G = (V, E)$ be a non-complete graph of order n and component number c . Then, for any $i = 1, 2, \dots, n$, $d_i \geq 0$ and equality holds if and only if one of the following cases happens:*

- (i). $i = n$;

- (ii). $1 \leq i \leq c - 2$;
- (iii). $i = c - 1$ and G does not have isolated vertices;
- (iv). $i = c = 1$ and G is C_n .

Proof. We first show that $d_i = 0$ in any one of the four cases above.

By (39), $d_n = \sum_{u \in V} [1 + (n - 1) \cdot 0] - n \cdot 1 = 0$.

It is known that for $1 \leq i \leq n$, $a_i(G) = 0$ if and only if $i < c$ (see [5, 13, 14]). Similarly, $a_i(G - u) = 0$ for all i with $1 \leq i < c - 1$ and all $u \in V$, and $a_{c-1}(G - u) = 0$ if u is not an isolated vertex of G . By (39), $d_i = 0$ for all i with $1 \leq i \leq c - 2$, and $d_{c-1} = 0$ when G does not have isolated vertices.

If G is C_n , then $a_1(G) = n - 1$, $a_0(G - u) = 0$ and $a_1(G - u) = 1$ for each $u \in V$, implying that $d_1 = 0$ by (39).

In the following, we will show that $d_i > 0$ when i does not belong to any one of the four cases.

If G has isolated vertices, then $a_{c-1}(G - u) > 0$ for any isolated vertex u of G and

$$\sum_{u \in V} a_{c-1}(G - u) = \sum_{\substack{u \in V \\ u \text{ isolated}}} a_{c-1}(G - u) > 0. \tag{40}$$

As $a_{c-1}(G) = 0$, by (39), we have $d_{c-1} > 0$ in this case. Now it remains to show that $d_i > 0$ holds for all i with $c \leq i \leq n - 1$, except when $i = c = 1$ and G is C_n .

For any $v \in V$, let $\mathcal{OP}'_{i,v}(V)$ be the set of ordered partitions $(V_1, \dots, V_i) \in \mathcal{OP}_{i,v}(V)$ with $V_1 = \{v\}$. As $\alpha(G[V_1], v) = 1$, for any i with $c \leq i \leq n$, by Theorem 8,

$$a_{i-1}(G - v) = \sum_{(V_1, \dots, V_i) \in \mathcal{OP}'_{i,v}(V)} \alpha(G[V_1], v) \prod_{j=2}^i \alpha(G[V_j], m_j), \tag{41}$$

where m_j is the minimum number in V_j for all $j = 2, \dots, i$.

Let s and v be distinct members in V . For any $V_1 \subseteq V - \{s\}$ with $v \in V_1$, let $\alpha(G[V_1 \cup \{s\}], v, s)$ be the number of those acyclic orientations of $G[V_1 \cup \{s\}]$ with v as the unique source and s as one sink. Then $\alpha(G[V_1 \cup \{s\}], v, s) \leq \alpha(G[V_1], v)$ holds, where the inequality is strict if and only if $G[V_1]$ is connected but $G[V_1 \cup \{s\}]$ is not. Observe that

$$\begin{aligned} a_i(G - s) &= \sum_{(V_1, \dots, V_i) \in \mathcal{OP}_{i,v}(V - \{s\})} \alpha(G[V_1], v) \prod_{j=2}^i \alpha(G[V_j], m_j) \\ &\geq \sum_{(V_1, \dots, V_i) \in \mathcal{OP}_{i,v}(V - \{s\})} \alpha(G[V_1 \cup \{s\}], v, s) \prod_{j=2}^i \alpha(G[V_j], m_j) \end{aligned} \tag{42}$$

$$= \sum_{(V'_1, \dots, V'_i) \in \mathcal{OP}_{i,v,s}(V)} \alpha(G[V'_1], v, s) \prod_{j=2}^i \alpha(G[V'_j], m_j), \tag{43}$$

where $\mathcal{OP}_{i,v,s}(V)$ is the set of ordered partitions $(V'_1, \dots, V'_i) \in \mathcal{OP}_{i,v}(V)$ with $s, v \in V'_1$. By the explanation above, inequality (42) is strict whenever $V - \{s\}$ has a partition V_1, V_2, \dots, V_i with $v \in V_1$ such that each $G[V_j]$ is connected for all $j = 1, 2, \dots, i$ but $G[V_1 \cup \{s\}]$ is not connected.

By (37), we have

$$\begin{aligned} na_i(G) &= \sum_{v \in V} \sum_{(V_1, \dots, V_i) \in \mathcal{OP}_{i,v}(V)} \alpha(G[V_1], v) \prod_{j=2}^i \alpha(G[V_j], m_j) \\ &= \sum_{v \in V} \sum_{(V_1, \dots, V_i) \in \mathcal{OP}'_{i,v}(V)} \alpha(G[V_1], v) \prod_{j=2}^i \alpha(G[V_j], m_j) \\ &\quad + \sum_{v \in V} \sum_{(V_1, \dots, V_i) \in \mathcal{OP}_{i,v}(V) - \mathcal{OP}'_{i,v}(V)} \alpha(G[V_1], v) \prod_{j=2}^i \alpha(G[V_j], m_j). \end{aligned} \quad (44)$$

By (41),

$$\sum_{v \in V} \sum_{(V_1, \dots, V_i) \in \mathcal{OP}'_{i,v}(V)} \alpha(G[V_1], v) \prod_{j=2}^i \alpha(G[V_j], m_j) = \sum_{v \in V} a_{i-1}(G - v), \quad (45)$$

and by (43),

$$\begin{aligned} &\sum_{v \in V} \sum_{(V_1, \dots, V_i) \in \mathcal{OP}_{i,v}(V) - \mathcal{OP}'_{i,v}(V)} \alpha(G[V_1], v) \prod_{j=2}^i \alpha(G[V_j], m_j) \\ &\leq \sum_{v \in V} \sum_{s \in V - \{v\}} \sum_{(V_1, \dots, V_i) \in \mathcal{OP}_{i,v,s}(V)} \alpha(G[V_1], v, s) \prod_{j=2}^i \alpha(G[V_j], m_j) \end{aligned} \quad (46)$$

$$\leq \sum_{v \in V} \sum_{s \in V - \{v\}} a_i(G - s) \quad (47)$$

$$= (n - 1) \sum_{v \in V} a_i(G - v), \quad (48)$$

where inequality (46) is strict if there exists $(V_1, \dots, V_i) \in \mathcal{OP}_{i,v}(V)$ for some $v \in V$ such that $G[V_j]$ is connected for all $j = 1, \dots, i$ and $G[V_1]$ has acyclic orientations with v as the unique source but with at least two sinks, and by (42) and (43), inequality (47) is strict if V can be partitioned into V_1, \dots, V_i such that $G[V_j]$ is connected for all $j = 2, \dots, i$ but $G[V_1]$ has exactly two components, one of which is an isolated vertex in $G[V_1]$.

As G is not complete, by Lemma 4 and the above explanation, the inequality of (48) is strict for all i with $c \leq i \leq n - 1$, except when $i = c = 1$ and G is C_n . Then, by (44), (45) and (48), we conclude that

$$d_i = \sum_{v \in V} [a_{i-1}(G - v) + (n - 1)a_i(G - v)] - na_i(G) > 0, \quad \forall c \leq i \leq n - 1, \quad (49)$$

except that $i = c = 1$ and G is C_n . Hence the proof is complete. \square

Now everything is ready for proving Theorems 3 and 4.

Proof of Theorem 3: Let G be a non-complete graph of order n . Recall (38) that

$$(-1)^n \left[(x - n + 1) \sum_{u \in V(G)} P(G - u, x) - nP(G, x) \right] = \sum_{i=1}^n (-1)^i d_i x^i. \quad (50)$$

By Proposition 4, we know that $d_i \geq 0$ for all i with $1 \leq i \leq n$ and $d_{n-1} > 0$. Thus $\sum_{i=1}^n (-1)^i d_i x^i > 0$ holds for all $x < 0$, which completes the proof of Theorem 3. \square

Proposition 5. *For any non-complete graph G , $\xi(G, x) > 0$ holds for all $x < 0$.*

Proof. We will prove this result by induction on the order n of G . When $n = 2$, the empty graph N_2 of order 2 is the only non-complete graph of order 2. As $P(N_2, x) = x^2$, by (23), we have

$$\xi(N_2, x) = (-1)^2 x^2 \left(\frac{1}{x} + \frac{1}{x-1} \right) + (-1)^3 2x = \frac{x}{x-1} > 0 \quad (51)$$

for all $x < 0$.

Assume that this result holds for any non-complete graph G of order less than n , where $n \geq 3$. Now let G be any non-complete graph of order n .

Case 1: G contains an isolated vertex u .

By the inductive assumption, $\xi(G - u, x) \geq 0$ holds for all $x < 0$, where equality holds when $G - u$ is a complete graph. By Lemma 1, $\xi(G, x) > 0$ holds for all $x < 0$.

Case 2: G has no isolated vertex.

By Theorem 3, (9) holds for all $x < 0$. Thus, for any $x < 0$, there exists some $u \in V(G)$ such that $(-1)^n (x - n + 1)P(G - u, x) + (-1)^{n+1}P(G, x) > 0$ holds. Then, by Lemma 3 and by the inductive assumption, $\xi(G, x) > 0$ holds for any $x < 0$.

Hence the result holds. \square

Proof of Theorem 4: It follows directly from Propositions 3 and 5. \square

6 Remarks and problems

First we give some remarks here.

- (i). Theorem 4 implies that for any non-complete graph G of order n , $\frac{P(G, x)}{P(K_n, x)}$ is strictly decreasing when $x < 0$.

(ii). Let G be a non-complete graph of order n and $P(G, x) = \sum_{i=1}^n (-1)^{n-i} a_i x^i$. Then $\epsilon(G) < \epsilon(K_n)$ implies that

$$\frac{a_1 + 2a_2 + \cdots + na_n}{a_1 + a_2 + \cdots + a_n} > 1 + \frac{1}{2} + \cdots + \frac{1}{n}. \quad (52)$$

(iii). When $x = -1$, Theorem 3 implies that for any graph G of order n ,

$$(-1)^{n-1} \sum_{u \in V} P(G - u, -1) \geq (-1)^n P(G, -1), \quad (53)$$

where the inequality holds if and only if G is complete. By Stanley's interpretation for $(-1)^n P(G, -1)$ in [18], the inequality above implies that for any graph $G = (V, E)$, the number of acyclic orientations of G is at most the total number of acyclic orientations of $G - u$ for all $u \in V$, where the equality holds if and only if G is complete.

Now we raise some problems for further study.

It is clear that for any graph G of order n ,

$$\frac{d}{dx} (\ln[(-1)^n P(G, x)]) = \frac{P'(G, x)}{P(G, x)} < 0 \quad (54)$$

holds for all $x < 0$. We surmise that this property holds for higher derivatives of the function $\ln[(-1)^n P(G, x)]$ in the interval $(-\infty, 0)$.

Conjecture 2. Let G be a graph of order n . Then $\frac{d^k}{dx^k} (\ln[(-1)^n P(G, x)]) < 0$ holds for all $k \geq 2$ and $x \in (-\infty, 0)$.

Observe that $\epsilon(G, x) = \frac{d}{dx} (\ln[(-1)^n P(G, x)])$. We believe that Theorems 2 and 4 can be extended to higher derivatives of the function $\ln[(-1)^n P(G, x)]$.

Conjecture 3. Let G be any non-complete graph of order n and Q be any chordal and proper spanning subgraph Q of G . Then

$$\frac{d^k}{dx^k} (\ln[(-1)^n P(Q, x)]) < \frac{d^k}{dx^k} (\ln[(-1)^n P(G, x)]) < \frac{d^k}{dx^k} (\ln[(-1)^n P(K_n, x)]) \quad (55)$$

holds for any integer $k \geq 2$ and all $x < 0$.

It is not difficult to show that Conjecture 2 holds for $G \cong K_n$. Thus the second inequality of Conjecture 3 implies Conjecture 2.

It is natural to extend the second part of Conjecture 1 (i.e., $\epsilon(G) < \epsilon(K_n)$ for any non-complete graph G of order n) to the inequality $\epsilon(G) \leq \epsilon(G')$ for any graph G' which contains G as a subgraph. However, this inequality is not always true. Let G_n denote

the graph obtained from the complete bipartite graph $K_{2,n}$ by adding a new edge joining the two vertices in the partite set of size 2. Lundow and Markström [11] stated that $\epsilon(K_{2,n}) > \epsilon(G_n)$ holds for all $n \geq 3$. In spite of this, we believe that for any non-complete graph G , we can add a new edge to G to obtain a graph G' with the property that $\epsilon(G) < \epsilon(G')$, as stated below.

Conjecture 4. For any non-complete graph G , there exist non-adjacent vertices u and v in G such that $\epsilon(G) < \epsilon(G + uv)$.

Obviously, Conjecture 4 implies $\epsilon(G) < \epsilon(K_n)$ for any non-complete graph G of order n (i.e., Theorem 4). Conjecture 4 is similar to but may be not equivalent to the following conjecture due to Lundow and Markström [11].

Conjecture 5 ([11]). For any 2-connected graph G , there exists an edge e in G such that $\epsilon(G - e) < \epsilon(G)$.

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