Passing a Proof Message: Student-Teacher Communication Through A Commognitive Lens

Weng Kin Ho
Nanyang Technological University
<wengkin.ho@nie.edu.sg>

Lim Seo Hong
Nanyang Technological University
<shlims@gmail.com>

Eng Guan Tay
Nanyang Technological University
<engguan.tay@nie.edu.sg>

Yew Hoong Leong
Nanyang Technological University
<yewhoong.leong@nie.edu.sg>

Teo Kok Ming
Nanyang Technological University
<kokming.teo@nie.edu.sg>

This study employs Sfard's (2008) socio-cultural theory of Commognition to analyse student teachers' thinking and communicating practices. Specifically, we investigate the effectiveness of the student teachers' communication of a particular mathematical proof with reference of the four features of the commognitive framework, i.e., word use, visual mediators, narrative and routines. In this paper, we can report on the routine of the discourse to analyse the quality of mathematical discourse in two situations of "Expert-to-Novice" and "Novice-to-Novice".

Introduction

The current research is an extension of a series of curriculum review meetings aimed at preparing student teachers of a certain teacher-training institute for mathematics teaching and further mathematics studies. The curriculum review processes consist of broad objectives of improving students' learning and teaching experiences; and developing disciplinary thinking of a mathematician by focusing more on reasoning skills. Specifically, the members of the review committee and teaching faculty pointed out that some of the mathematics courses were overly focussed on assessment and examinations. As a result, students learnt by memorizing the content with little understanding of either the course content or its real objectives. As a result of excessive assessment, weaker students did not gain positive learning and teaching experiences.

To enhance student teachers' experiences, classroom discourse is essential for them to think and communicate complex mathematical concepts. The participatory nature of classroom interaction facilitates rich learning opportunities for students to identify various aspects of learning and teaching challenges; and to be able to apply the necessary skills for improvement. More importantly, in a discursive classroom, a lecturer models and explains his thought processes when analysing and completing an activity. This dialogical method prompts students to gain insights on thinking, communicating, and learning in a mathematics classroom. Subsequently, they might be more likely to establish the connections between what they have learnt in class and how to apply these valuable skills in their future teaching.

In the present study, the research team employs Sfard's (2008) socio-cultural theory of Commognition to analyse student teachers' thinking and communicating practices. This research investigates the effectiveness of the student teachers' communication of a particular

2019. In xxxx (Eds.). Mathematics Education Research: Impacting Practice *(Proceedings of the 42nd annual conference of the Mathematics Education Research Group of Australasia)* pp. xxxx. Perth: MERGA.

mathematical proof with reference to the four features of the commognitive framework, i.e., word use, visual mediators, narrative and routines. In this paper, we can only report on the routine portion of the discourse with an overlay of the 'rigorous-explanatory' dichotomy of proof discourse. Crucially, we identify the distinctive routines present in the "Expert-to-Novice" and "Novice-to-Novice" discourses respectively.

Theoretical Framework

Sfard introduces the conceptual framework of Commognition by highlighting that thinking is dialogical in nature. Sfard redefines thinking as communicating with oneself and others (Sfard, 2008). Thinking (individual cognition) and communicating (interpersonal communication) are two parts of the same entity. Therefore, Commognition refers to the interaction of cognition and communication. This paradigm frames learning as a unified body of cognition and communication. Sfard's conceptualization of Commognition has been largely influenced by Wittgenstein's and Vygotsky's theories. Sfard (2008) points out that for Wittgenstein, meaning is neither a thing in the world nor a private entity in one's mind, rather meaning is an aspect of human discursive activity that is fully investigable. Thus, people use words not only to reflect the world but also to create meanings through language with logical structures. Following this line of argument, a discourse can thus be operationalized by observing how people use words; and rules can therefore be teased out according to what they do.

The above position found similar expression in Vygotsky's work. Vygotsky illustrates that studying thinking by attending to words and thoughts as separate entities would be like trying to find out the properties of water by looking at hydrogen and oxygen (Sfard, 2008). According to Vygotsky, knowledge, concept, and higher mental function are culturally produced. Constantly, these different elements undergo modifications because of collective human efforts. Vygotsky views language as the instrument to develop thought. The confluence of Wittgenstein's and Vygotsky's theories points out that learning of mathematics can be facilitated by meanings and language in people's discourse.

Essentially, the commognitive framework is based upon the social dimensions to capture social participation patterns, mathematical discourse, and identity narratives (Heyd-Metzuyanim & Sfard, 2012; Sfard, 2008; Sfard & Prusak, 2005). Hence, learning is not located in the heads or outside of the individual; it is a result of relationships between an individual and a social world. This discursive perspective espouses the Socio-Cultural tenet in which learning occurs in the situational, cultural, and historical milieu by firmly emphasizing the view of thinking as a type of communication (Kieran, Forman & Sfard, 2002).

Word Use, Visual Mediators, Narrative and Routines

To analyze mathematics and changes in thinking, Sfard proposes four features to categorize mathematical discourse: word use, visual mediators, narrative, and routines. Firstly, word use refers to the keywords employed by the participants of the discourse and what they mean by these words. Word use is an 'all important matter because … it is responsible for what the user is able to say about (and thus to see in) the world (Sfard, 2008, p. 133). For instance, words like 'cardinality', 'set', and 'element' have specific meanings to experts, and yet novices may not share the same understanding.

Secondly, visual mediators refer to non-verbal means of communication. Examples of these are diagrams, symbolic artefacts, mathematical algebraic notation, and even gestures. Thirdly, narrative refers to spoken or written mathematical statements, such as "$\mathbb{Q}^n$ is countable”. To either agree or disagree with the statement, participants can argue through word use, visual mediators, and routines. In mathematical discourse, the endorsed narratives are definitions, proofs, and theorems in the mathematics community. Finally, routines refer to a collection of meta-rules that describe a repetitive discursive action. Such repetitive patterns occur in the use of words and visual mediators, or in the process of creating and substantiating narratives. Typically, a routine is time-initiated by an applicability condition, time-culminated by a closure (these form the ‘when’ of a routine), and implemented via a procedure (this forms the ‘how’ of a routine). In this paper, we focus on three kinds of routines, namely, exploration, deeds, and rituals. Exploration results in the creation of mathematical theory – the endorsed narrative of a mathematical result is produced. These narratives need to be constructed, substantiated and recalled in the mathematical discourse. Deeds refer to routines that involve an action that produces a physical change in objects in the environment; examples include physical entities and even written work. Rituals are a distinguished routine whose goal is neither to produce endorsed narratives nor physical changes, but rather to establish a social bond between the interlocutors.

Methods

We now illustrate the methods employed in our present study.

Participants

Two groups of undergraduate year one student teachers participated in this research: three male and three female students (N = 6). Group A consisted of one male and two females, while Group B involved one female and two males. During the first lesson of a certain Mathematical Problem-Solving course, consent forms were given to all students to obtain their consent and to inform them of the research purpose. To report their responses, pseudonyms (RA, RB: same researcher in both groups; A1-3: Group A participants; B1-3: Group B participants) would be used.

Procedures and Experimental Design

To have an insight into the student teachers’ thinking and communicating processes, the participants were to explain to their peers the main theorem that “The cartesian product $\mathbb{Q}^n$ of n copies of the set $\mathbb{Q}$ of rational numbers is countable”. To begin, the researcher explained the content of the activity to the first participant. An information sheet containing definitions regarding countability and a proof of the countability of the set $\mathbb{Q}$ of rational numbers was then given. The student was given up to five minutes to re-acquaint with the materials which had already been taught in class. Then, the researcher showed the student how to prove the theorem. Next, the first participant repeated the procedure to the second participant. The first participant could use the materials written by the researcher as reference, but was not to show them directly to the second participant; instead he/she was to use his/her own words or writing in the explanation. This was repeated from the second participant to the third, and finally from the third participant to the researcher. Each participant took approximately half an hour to learn and then to teach the mathematical proof subsequently. After the third participant completed the last round, together with the researcher as a group they discussed the learning and teaching processes that had just taken place. A similar procedure applied to the second group of participants but they had the group discussion at a different day due to time constraint. During the post-activity discussions for both groups, the participants were given a page of discussion questions around which they reflected on the communication in relation to the four features of the commognitive framework, i.e., word use, visual mediators, narratives, and routines.

The main theorem and its proof form the vehicular body of mathematical narrative to be endorsed by both the 'teacher' and the 'learner' within each session of communication. The proof comprises three parts: (i) the cardinality of $\mathbb{Q}$ is at most that of $\mathbb{Q}^n$ by appealing to the embedding that maps $x \in \mathbb{Q}$ to $(x, 0, \ldots, 0) \in \mathbb{Q}^n$, (ii) the cardinality of $\mathbb{Q}^n$ is at most that of $\mathbb{N}$ (and hence that of $\mathbb{Q}$, as justified by the information sheet provided) by relying on the Gödel enumeration, and (iii) the version of 'Squeeze Theorem' concerning cardinality of sets, i.e., if the cardinality of $\mathbb{Q}$ is at most that of $\mathbb{Q}^n$ and vice versa, then both have the same cardinality, and hence $\mathbb{Q}^n$ is countable. Parts (i) and (ii) appeal to 'common sense', and so they are considered easy; only part (iii) is regarded as the 'hard' part of the proof. We shall be zooming in on (iii) through the commognitive lens.

Data Collection

The Activity, "The cartesian product of *n* copies of the set $\mathbb{Q}$ of rational numbers is countable", is the main instrument for data collection. All the meeting sessions were audio-recorded. The participants' written artefacts were scanned and the originals were returned to them. Discussion questions below were used at the end of the Activity in order to gain a more accurate understanding of participants' perceptions in relation to the commognitive framework during the activity:

- Were you careful about the use of words and their meanings? As student; as teacher?
- Did you feel the need to use 'visual mediators' such as writing symbols, diagrams, using gestures? As student; as teacher?
- Did you sense that the communication had to follow a certain type of protocol, such as of logic and of a mathematical environment of definitions and theorems? As student; as teacher?
- Did you find yourself going through some accepted routines, such as calculating numbers, doing problem solving (Polya and heuristics)? As student; as teacher?

Data analysis and Findings

The transcriptions for all the discourses were coded, marking clearly the word use, visual mediators, narratives and routines that were employed by each interlocutor. Here we only analyse the occurrences of the three kinds of routines (exploration, deeds, rituals), i.e., how and when they were implemented in the 'Expert-to-Novice' and 'Novice-to-Novice' settings.

Expert-to-Novice

In this setting, the researcher consistently engages in *explorations* with the student teacher A1 (resp., B1) which move them towards constructing and understanding the intended mathematical proof. For example, construction of narratives are aided with realization procedures such as the one employed by the researcher in clarifying B1's misconception that $\mathbb{Q} \times \mathbb{Q} \times \ldots \times \mathbb{Q}$ to be the ordinary multiplication of rational numbers:

B1: Is $\mathbb{Q}^n = \mathbb{Q} \times \mathbb{Q} \times \ldots \times \mathbb{Q}$?
R: Hum [agreeing]
B1: I choose any element from $\mathbb{Q}$ and multiply with another?

R: The set is a collection of all these things.
B1: Ohh…so, I can say that each *n* has different $\mathbb{Q}$ is it?
R: You can think of it that way. There are different ways of thinking, the easiest way is to think of your familiar knowledge, which is your cartesian plane, that's where cartesian product comes about. Your cartesian plane is the coordinates X and Y right? What can be X?

By inviting B1 to 'think of … familiar knowledge', the researcher aids him/her to invoke previously endorsed narratives (i.e., meaning of cartesian product of two sets) through *recalling*. Elsewhere, the researcher also reminds another participant A1 about the definition of an injective function.

R: I am going to get an injection … it is called injection … "one-one function" called an injection. I am going to get a "one-one" function from $\mathbb{Q}^n$ to the natural numbers.

Additionally, at several junctures, the researcher makes a conscientious effort to enforce routines of *substantiation* of narratives; for instance, a rigorous proof is given whenever the participant is not convinced of an argument/claim:

R: No, only this one [at this point, they were using symbols on papers to aid their communication], this one comes from the idea that obviously this guy is so much bigger than this guy
B1: Hum…ooh…so this is bigger but (few ahh) [unclear]
R: But I can prove it rigorous, not happy with me, I'll prove it rigorously for you ok?
B1: ok

Already pointed out in Example 6 in (Sfard 2008, p. 238), adults' invitation for explorations may prompt a child to perform a deed. Interestingly, a similar observation is made of student teachers (who are themselves adults, not children) in "Expert-to-Novice" settings. In the following episode, a concrete example of a 5-tuple of rational numbers is assigned its Gödel enumeration intended by the researcher as an exploration for the student teacher.

R: So now it is $\mathbb{Q}^n$, just use $\mathbb{Q}^5$. Easy to see. OK? Because all you need is an example.
A1: Emm…
R: So just give an element of $\mathbb{Q}^5$.
A1: $\left(\frac{1}{2}, 1, -1, 0, -\frac{2}{3}\right)$ [some exchanges here…, deciding which number to give in $\mathbb{Q}^5$] Wait ha…
R: which number will it go to? That's the question. [invitation for exploration]
[For the next 10 minutes, A1 performs the concrete action of Godel coding as an individualized activity that produces the symbolic change of transforming the given rational tuple into a natural number.]
R: Ya… greater than 0, ya. Then it is taken care here already. In this block of 3, it is taken care here already.
Once you determine the sign, then your mind relax.
You just look at the a and the b.
A1: OOO!!!! hmmm… ok.
I am…ok, sorry. I got it again, ok. I got it. Come back come back… -1
R: -1
A1: So err… 17,
R: 17
A1: to the power of 1
R: Yes!
A1: 19 to the power of 0.
R: No. -1 is written as "-1" over 1.
A1: So ok, 1 again.
R: 1 again, absolute value.
A1: 23 to the power of …1.
R:b …ok? Done.

The subsequent discourse, for the participant A1, is centred entirely around the performance of the deed, rather than the culmination with an endorsed narrative of getting the desired injection from $\mathbb{Q}^n$ to $\mathbb{N}$.

In a few occasions, it is observed that participant B1 is responding to the researcher's prompt for exploration by performing, instead, a *ritual* of affirming that he/she understands even though it is not the case.

> R: Hum [indication of approval]. This element goes to this huge number, this huge number natural, integer, right? So, it goes into here. If the two elements which are different, would they go to two different numbers?
> B1: Yes, yes. Yeah [agreeing with R's questions].
> R: That is 1-1, already.
> B1: Ohh…okay
> R: You see that?
> B1: Yes
> R: It is clear? [laughter] Is it clear?
> B1: Yes. It is lah. [unsure]
> R: Later, you have to explain it to your friend?

In seeking approval and maintaining a level of 'social acceptance' from the researcher, the participant B1 chose to respond positively to every question raised by the researcher to the extent that the researcher starts to sense that B1 does not seem to understand completely. At this juncture, the researcher reminds B1 that his/her duty will be to explain the proof clearly to the next person. The participant then detaches himself/herself from the ritual and requests for a formal substantiation.

Novice-to-Novice

Unlike the scenario of "Expert-to-Novice" discourse which is dominated by explorations created by the researcher as an invitation to construct the intended narratives, the situation of "Novice-to-Novice" discourse is mainly situated by deeds and rituals. Many of the student teachers' "teaching" aims at conveying accurately the method of Godel enumeration to the next person, rather than on the bigger picture of proving the equality of cardinalities of $\mathbb{Q}^n$ and $\mathbb{N}$.

> B1: Now, we have to prove this.
> B2: How to prove this?
> B1: Yeah, I'm gonna show you how to prove this, later, you show B3 how to prove this.

The pre-occupation with 'getting things right' took place in the earlier part of most the conversations. However, it is noted that some participants switch from the routine of ritual to that of genuine exploration, and even requests for such.

> B1: So this is true, but this one you know how to explain right, if B3 asks you.
> B2: But there is no like…is just smaller…
> B1: You can prove…there…you can prove, you want to see the proof [sensing so]?
> B2: I want to see the proof!

Conclusion

Throughout the "Expert-to-Novice" sessions, the researcher makes a conscientious effort to engage the learner by presenting several opportunities of *exploration*, aided frequently by visual mediators such as diagrams (see Figure 1).

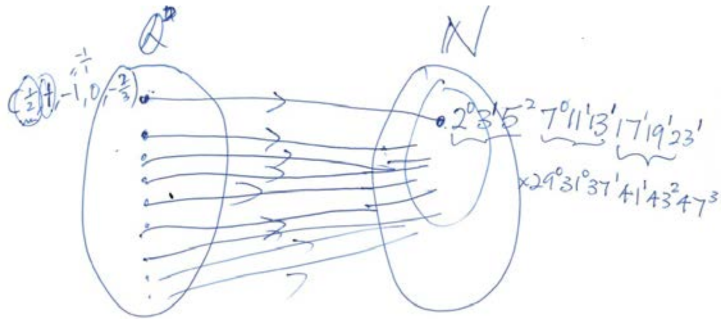

Figure 1. Visual Mediator: Injective function from $\mathbb{Q}^n$ *to* $\mathbb{N}$

One important routine the researcher uses is a consistent heightening of the student's senses to the claim that the cardinality of $\mathbb{Q}^n$ is at most that of the set of natural numbers. In one episode, we have:

> R: Now surprisingly, this set is *countable.*
> A1: [non-verbal response]

Another episode reveals:

> R: That's why, is very powerful, can go to a million, but the entries here must only be Q. Is what you call some kind of, is not all the real numbers, is a lattice [spelling?], is discrete. I can only take those numbers, I am not allowed to have pi, 1, 1, 1, 1, not allowed. The cartesian one, any real number will do. Okay, a huge number, there are many points like that, but surprisingly I can get them all in natural numbers.
> B1: ok [followed by silence]

However, none of the "Novice-to-Novice" discourses conveys this 'surprise' element, due to the preoccupation of relaying the correct details of the Gödel enumeration. Our analysis of the routines held out by the novices shows that they are mostly engaged in routines of deeds and rituals, with only a slight shift towards exploration as they realize they have the responsibility of conveying the correct proof back to the researcher eventually.

Looking through the commognitive lens allows us to appreciate better how experts and novices engage in a mathematical discourse. We have seen that experts and novices employ quite different repetitive patterns in their discursive actions. One possible lesson drawn from this study could be that communication of mathematical ideas and proofs may serve as an auxiliary channel of assessment of understanding of mathematics, in addition to the traditional assessment modes. Apprenticeship for novices under the supervision of experts can be structured to train novices in a more engaging and productive mathematical discourse.

References

Heyd-Metzuyanim, & Sfard, A. (2012). Identity struggles in the mathematics classroom: On learning mathematics as an interplay of mathematizing and identifying. *International Journal of Educational Research*, 51-52, 128-145.

Kieran, C., Forman, E., & Sfard, A. (Eds.). (2002). *Learning discourse: Discursive approach to research in mathematics education.* Dordrecht: Kluwer Academic.

Sfard, A. (2008). *Thinking as communicating. Human development, the growth of discourse, and mathematizing.* New York, NY: Cambridge University Press.

Sfard, A., & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. *Educational Researcher*, 34, 14-22.