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As Long As the Drawing is Logical, Size Does Not Matter

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The use of letters as a variable is one of the obstacles faced by students as they make the transition from arithmetic to letter-symbolic algebra. Primary students in Singapore are in a unique position as they are presented with opportunities, right across their primary years, to work with less abstract representations of numbers. These students are taught to use a set of proportionately sized rectangles to model arithmetic as well as algebra word problems. fMRI studies show that adults used similar processes in generating the rectangles and letters used to solve algebra-word problems. How do primary students perceive these model representations? Ten primary 5 students were interviewed on a one-to-one basis after they had completed a series of mathematics tasks. The written and interview data suggest that these students' familiarity with the model method had enabled them to construct a new meaning of how numbers could be represented. This finding may offer educators an alternative route to facilitate the transition from arithmetic to letter-symbolic algebra.

Although many students enjoy arithmetic, many find the transition from arithmetic to algebra challenging (Kieran, 1992). Students may not realize it but the learning of algebra "starts as the art of manipulating sums, products, and powers of numbers. The rules for these manipulations hold for all numbers, so the manipulations may be carried out with letters standing for the numbers" (MacLane & Birkhoff, 1993, p. 1). Hence the most obvious change students face moving from arithmetic to algebra is "the latter's use of letters to represent values" (Booth, 1988, p. 26). Although in arithmetic, letters are used to represent whole words, for example h for hours and s for seconds, in algebra the same letters could be used to represent number of hours and number of seconds respectively. That students find the change in representational system problematic is reported in a study by Küchemann (1981). He found most 13 – 15 year olds were unable to work consistently with items that required treating letters as unknown numbers. In Piagetian terms, these students were still operating at the concrete-operational stage. What was more disturbing was the study by Coady and Pegg (1991) who found that tertiary students were not much better with the Küchemann items than the secondary school students who participated in the original study. This suggests that much of what students do in school -- the structural aspect of algebra, where letters are treated as "arbitrary marks on paper" (Usiskin, 1988, p. 17)

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that had to be manipulated -- did not assist students' understanding with the meaning of letters as variables.

It could be that the transition from numbers to letters was too big a leap for secondary students to make if even college students were still operating at the concrete-operational stage (Küchemann, 1981). Would using a pictorial and hence less abstract representation for numbers better prepare students to use letters as representations for numbers? The Singapore mathematics curriculum offers us a chance to investigate this possibility. Problem solving is the core of the Singapore mathematics curriculum (Curriculum Planning and Development Division, CPDD, 2006). To engage primary students in the problem-solving process, Pólya's (1973) four stages is used as a guide to support and develop various problem-solving heuristics. In Singapore, the heuristic "draw a diagram" (CPDD, 2006, p. 14) , better known by teachers and students alike as the model method, is particularly popular because it can be used to solve arithmetic as well as algebra word problems. In arithmetic word problems, rectangles are used to represent specific numerical inputs. Rectangles are representations of variables when the model method is used to solve algebra word problems. This exposure to a less abstract representation may prepare students to work with letters as variables; hence the pictorial yet concrete representation could serve as a bridge between working with numbers and the very abstract nature of letter-symbolic algebra.

Arithmetic and algebra word problems and their related model solutions

We define arithmetic word problems as those where specific numerical inputs that are needed to solve for the unknown output are given. Questions in figures 1 and 2 are examples of arithmetic word problems presented in a local textbook (Collars, Koay, Lee & Tan, 2007). Given the numerical inputs, the objective of each of these questions is to find the output sum. The arithmetic word problem can be represented structurally as $a + b = x$, where a and b are the inputs and x, the unknown output to be evaluated. The example in Figure 1 has addends smaller than 10, while those in Figure 2 are bigger than 1000.

What is the sum of 4 and 6?

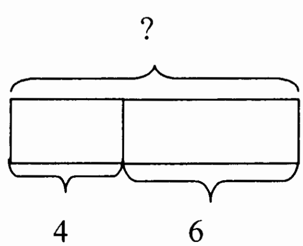


Figure 1. Using the Model Method to Represent an Arithmetic Problem Where the Inputs Are Less than 10. Source: Collars, Koay, Lee & Tan, 2007, 3A, p. 27

While the objective of arithmetic word problems is to ascertain the output given the numerical inputs, the objective of algebra word problems is to use the given numerical output to determine specific input. Questions in figures 3 and 4 are two such exam

On Saturday, 1050 people went on a cruise.
On Sunday 1608 people went on the cruise.
How many people went on the cruise on both days?

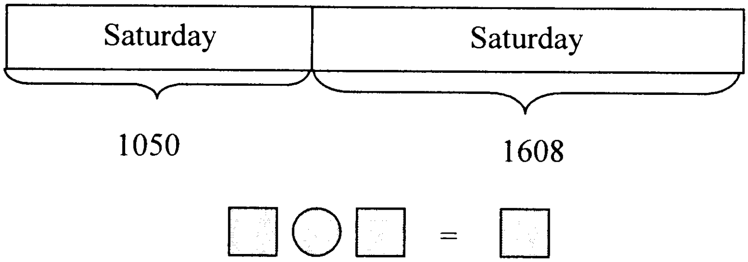


Figure 2. Cruise Problem, an Example of an Arithmetic Word Problem and its Accompanying Model Solution. In This Example the Inputs Are More Than 1000. The Attendance on Each Day Represents the Given Input And The Total Number of Passengers as the Unknown Output. Source: Collars, Koay, Lee & Tan, 2007, 3A, p. 43

ples. Structurally these algebra word problems can be represented as $a \pm x = b$ and $ax = b$ respectively, and in each case, the input x has to be evaluated. Filloy and Rojano (1989) would not classify such problems as algebraic as the unknown variable appears only on one side of the equation, nevertheless such problems are commonly found in the algebra section of secondary textbooks (e.g. Teh & Looi, 1997). For this reason, we will classify them as such.

There were 2659 visitors at Orchid Gardens.
447 of them were adults and the rest were children.
How many children visited Orchid Gardens?

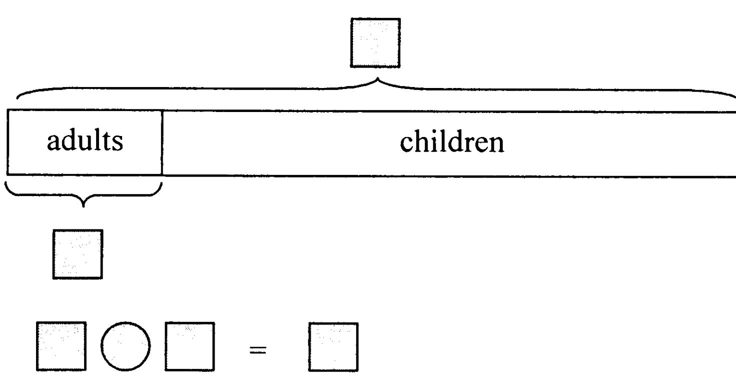
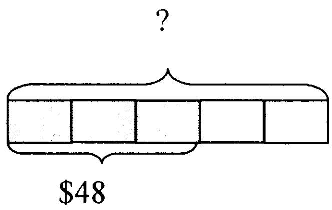


Figure 3. Orchid Problem, an Example of Algebra Word Problem and its Accompanying Model Solution. The Total Number of Visitors Represents the Known Output and The Number of Each Group Visitors, the Unknown Input. Source: Collars, Koay, Lee & Tan, 2007, 3A, p. 43

Edwin spent $\frac{3}{5}$ of his money. He spent $48. How much money had he at first?



3 units = $48

1 unit = $?

5 units = $?

He had $___ at first.

Using letter-symbolic method

Let x be the initial amount of money.

$$\frac{3}{5}x = \$48$$

$$x = \$48 \times \frac{5}{3}$$

$$x = \$80$$

Figure 4. Using the Model Method to Solve Problems Involving Concepts of Proportional Reasoning. The Left Panel Shows the Model Method Approach While Letter-Symbolic Method in the R/ight. Source: Collars, Koay, Lee, Ong & Tan, 2007, 4A, p. 65

THE MODEL METHOD

The Singapore mathematics curriculum introduces concepts of algebra only in primary six, the final year of primary school (Curriculum Planning and Development Division, 2006). Even then, students are introduced to the representation aspect of algebra: how letters can be used to represent variables and the structural aspects of school algebra (Usiskin, 1988). However, even primary school students are introduced to word problems requiring algebraic thinking.

Primary school students are taught to solve these questions using a variety of heuristics, the most popular of which is the model method which involves the use of visual or concrete representation for numbers. Rectangles of proportional lengths are used to represent the numerical inputs of problems like those in Figures 1 and 2. For arithmetic word problems, the length of each rectangle reflects the size of the number, the longer the rectangle, the bigger the number. Hence, in Figure 1, a longer rectangle is used to represent the number six and a shorter one for the number four since six is bigger than four. The same part-whole concept is applied when a model is used to represent the arithmetic word problem in Figure 2. In the orchid problem, the number of adults is smaller than the total number of visitors. By mentally comparing the number of children against the number of adults, the length of the rectangle used to represent the number of children is estimated to be proportionally longer than that used to represent the number of adults. The model method is also used to solve problems where concepts of proportional reasoning are applied. For example the model method solution provided to the question in Figure 4 (Collars, Koay, Lee, Ong & Tan, 2007) shows how one whole rectangle is divided into five equal units. In this example, a question presented as a fraction is transformed into one where whole numbers are used to solve the problem. Here the value represented by one unit is assumed to be the unknown and can be found without applying fraction related procedures.

Knowledge of the model method and other problem solving heuristics has meant that students without recourse to letter-symbolic algebra may successfully solve such problems. Primary students are more familiar with the model method since it is used to solve simple arithmetic problems. In arithmetic word problems, the rectangles represent known numbers where the lengths of the rectangles are proportional to the size of the numbers. However, in algebra word problems, the rectangles, known as units, represent variables to be evaluated; the lengths of the rectangles are necessarily arbitrary.

When Primary 1 and Primary 2 students used the model method to solve arithmetic word problems, the rectangles represent specific numbers. Beginning in Primary 3, algebra and arithmetic word problems are taught concurrently. To assimilate the new information students who use the model method to solve algebra word problems will have to extend their understanding of models as concrete objects -- as experienced in arithmetic word problems -- to one in which they behave as variables. In algebra word problems, the length of rectangles used in models is arbitrary. Rectangles of the same lengths can be used to represent numbers of any size. Do movements from arithmetic to algebra word problems and back again to arithmetic word problems, encourage students to be flexible in their thinking and suspend their perception of the role of rectangle as fixed concrete objects?

The study reported in this paper is part of a larger project investigating the cognitive demands of using the model method and letter-symbolic algebra to solve algebra word problems in primary as well as secondary school. Earlier studies using functional magnetic resonance imaging (fMRI) showed that adults used similar cognitive processes to generate equations and model representations of problems. The main difference between the two heuristics is that the generation of equation had higher attentional demands. There was also evidence to suggest that the generation of equation drew more heavily on procedural production. If adults use similar processes to generate models as do equations, how do primary students generate or perceive the rectangles used in the model method? Do they treat the rectangles as concrete objects where each rectangle is perceived as a representative for a specific number? Or do they treat these rectangles as abstract vessels, capable of containing any quantity? To answer these questions, ten Primary 5 students were interviewed on a one-to-one basis after they had completed a paper and pencil test.

METHOD

Participants

The objective of this study is to gain insights into students' perception of the rectangles used in the model method. Hence only students who were proficient in the model method were selected to participate in this study. Using a clinical interview approach, we conducted detailed discussions with ten Primary 5 (i.e. Grade 5) students (seven boys) attending a neighborhood school. Neighborhood' schools are non-selective schools situated in housing estates. Parents choose these schools because of their proximity to their homes.

Procedure

Students completed a paper and pencil test, which was then followed immediately by a semi-structured face to face interview where students were asked for their perceptions of the role of rectangles used in the model method.

The instrument

The written test consisted of two sections with production and validation tasks.

Production tasks. These task were divided into four sections, A – D, with six questions per section, and 24 questions in total. Each section consisted of similar curriculum based questions with numbers confined to a particular quantitative range. Table 1 shows the range of output for the 6 items per section. Across sections, the output changes in size. Table 2 lists an example of one question from each section. Appendix A lists the six questions presented in section D.

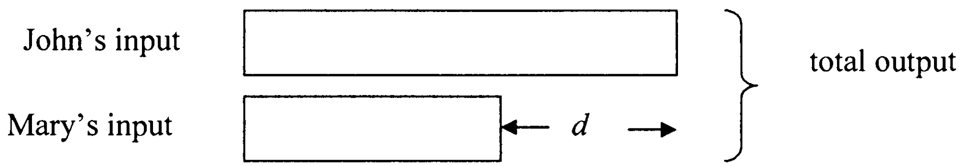
Table 1
Range of Output for Each of the Four Sections

section	A	B	C	D
range of output	0 - 9	10 - 99	100-1 000	1 000-10 000

Table 2
An Example of the Questions from Each Section, A - D

Section A	Mary and John have 8 marbles altogether. John has 6 marbles more than Mary. How many marbles has Mary?
Section B	Mary and John have 80 marbles altogether. John has 60 marbles more than Mary. How many marbles has Mary?
Section C	Mary and John has 800 marbles altogether. John has 600 marbles more than Mary. How many marbles has Mary?
Section D	Mary and John have 8 000 marbles altogether. John has 6 000 marbles more than Mary. How many marbles has Mary?

Except for the changing total numerical output and the difference in marbles between Mary and John, all the problems had the same structure. This was to ensure that the students' performance was not confounded by other factors, such as changing context or different semantic structure. An earlier study which explored how 151 primary 5 students from five different schools used the model method to solve algebra word problems, suggested that the following model drawing was most likely for question of this particular structure (Ng & Lee, 2005).



Within each section, the total output for the first four items of each section is kept constant (for section D, the output is 8000), John and Mary would each have different number of marbles when the difference *d* is changed. If concrete thinkers who apply proportional reasoning to draw rectangles to represent John and Mary's inputs, they would likely draw longer rectangles for bigger numbers. For these students, their model drawings for the first four items would have rectangles of varying lengths. Abstract thinkers were expected to have abstracted the role of rectangles as variables and to realize that the size of the rectangles did not matter. They were expected to draw rectangles of about the same size for any given numerical output.

Validation tasks. In the production tasks, the participants had to produce their own solutions. However, their solutions may reflect only what they are taught and not what they truly understand. To pre-empt erroneous conclusions based on such disconnections, we presented students with two types of validation tasks.

Type A validation tasks tested how strictly students adhered to proportional reasoning in the construction of models. All three type A validation tasks had the same output but different difference d, where d ranged from single digit to a three digit numbers. In this task, two hypothetical students, P and Q were portrayed as having given different sized model drawing solutions to the same item. In each case, student P applied proportional reasoning in the construction of the model drawing while student Q suspended proportional reasoning but constructed a model which was logically correct. Students were asked to decide whether the solutions were acceptable and to justify their choice of solutions. Table 3 shows one example of type A validation task where the hypothetical student P applied proportional reasoning when constructing the models. The rectangles representing the unknown input is about twice the length of the difference: 30 marbles. Hypothetical student Q's solution was logically correct; however it did not reflect the relationship between the difference and the unknown unit. The length of the rectangle representing the unknown value was about six times the length of the rectangle representing the difference. Hence this would suggest, erroneously, that Mary has 180 marbles.

There were three type B questions, with each set comprising two questions. Both these items had the same structure differing only in the output and the difference d. The solutions to the two questions by hypothetical student P, an abstract thinker, was presented first. Student P drew exactly the same models for the two questions. To focus students' attention on each set of solution, the solution of hypothetical student Q, a concrete thinker was presented on the next page (see Table 4). The hypothetical concrete student presented model drawings where the rectangles were proportional in length to each other and also reflected the size of the output, the bigger the output, the longer the rectangles. Students were asked to consider the accuracy of our hypothetical problem solvers.

Table 3
Type A Questions

"Here are some word problems. Student P and Student Q drew the models for these problems. You have to pick if Student P is correct, or Student Q is correct, or if both are correct. You can do this by checking the box next to the options.

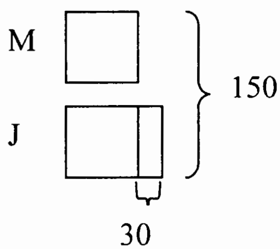
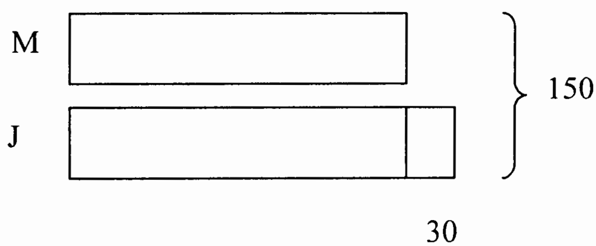
Explain your choice of answer by typing in the space provided.

1. Mary has some marbles.

   John has 30 marbles more than Mary.

   They have 150 marbles altogether.

   How many marbles has Mary?

Student P drew:	Student Q drew:
	

Semi-structured face to face interview

Immediately after the written test, we interviewed the students using a list of basic questions as a guide (see Appendices B, C and D). When necessary, we paraphrased the questions, omitted ones that were answered by students' earlier responses, and asked probing questions when we did not understand students' answers. The students took about 30 minutes to complete both the production and validation tasks. The students were then interviewed on a one-to-one basis. The shortest interview took 30 minutes to complete, the longest, an hour.

Analysis

Data consisted of students' written responses to the production and validation tasks as well as their verbal responses to the interview questions. First an across-participant analysis was conducted to identify whether students presented different model solutions to items within each set. Second, a within-participant analysis was conducted to determine differences in participants' responses across items with different outputs. This was to identify whether students were concrete or abstract thinkers. Third, students' responses to validation tasks were analyzed to identify for consistency with the production tasks. How did the students explain the solutions provided by the hypothetical abstract and concrete thinkers? How did the abstract students reconcile the solutions provided by the hypothetical concrete students? Similarly, how did concrete students reconcile the solutions provided by the hypothetical abstract thinkers?

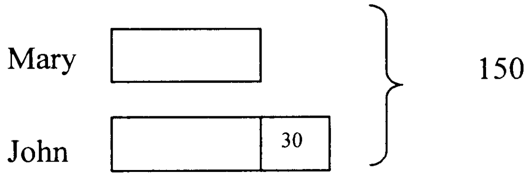
Table 4
Type B Questions

Student P and Student Q drew models for the 2 questions below.

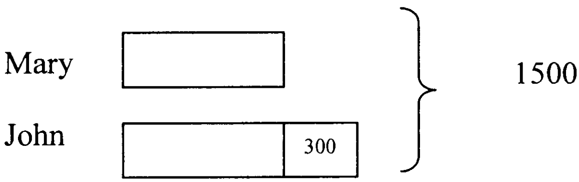
Q.1	Q.2
Mary has some marbles. John has 30 marbles more than Mary. They have 150 marbles altogether. How many marbles has Mary?	Mary has some marbles. John has 300 marbles more than Mary. They have 1500 marbles altogether. How many marbles has Mary?

Student P drew:

Q1

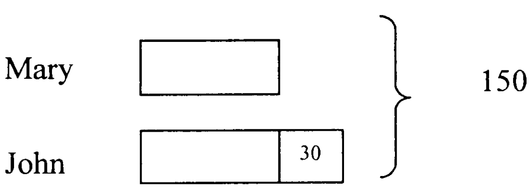


Q.2

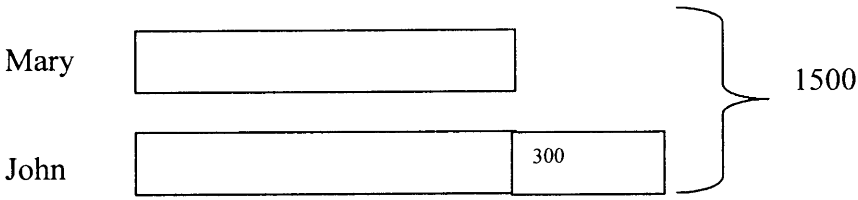


Student Q drew: (presented on a separate page

Q.1



Q.2



RESULTS

Students' written responses showed that they were able to use some form of the model method to solve the word problems. We identified two groups - the compareson model group and rectangles in a row group. Within each group we further distinguished the potentially abstract thinkers from the concrete-abstract thinkers. There were no pure concrete thinkers.

Comparison model group. Eight students drew the comparison model to solve items in sets A – D. Five students (AT1 – AT5) used single un-partitioned rectangle of equal length to represent the input - the number of marbles held by Mary and John and another rectangle to represent the difference. For three other students, AT6, AT7 and AT8, the comparison models they drew for items in Set A were different from those constructed for items in sets B – D. These three students represented the number of marbles held by Mary and John by partitioning a single rectangle into smaller rectangles, one rectangle per marble. Because numbers in set A were below 10, these three students worked out the solution to the items in set A as they drew the exact number of marbles for Mary and John. Figure 5 shows an example of each. The left panel shows how individual rectangles of the same size were used to represent each marble while the right shows how un-partitioned rectangles were used to represent John and Mary's marbles. However while AT 6 and AT7 changed to un-partitioned rectangles to represent the inputs for items B – D, AT8 drew different models for items in sets B – D (see Figure 6). Student AT8 used un-partitioned rectangles to represent the unknown input. However the difference was represented differently. If the difference was small (under 10), AT8 represented the difference precisely, a rectangle for each number. However when the difference was big (above 10), a partly partitioned left and right end of the difference rectangle with dots in between to suggest that an indeterminate number of rectangles was used to represent the difference.

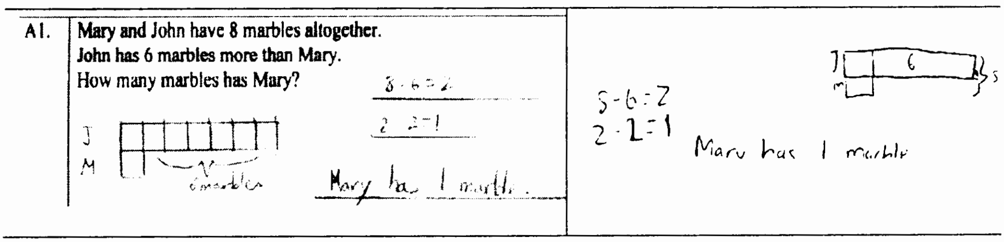


Figure 5. Left Panel Shows How a Student Partitioned Rectangles, Where Each Smaller Rectangle Represents a Single Marble. Right Panel Shows How Student AT 8 Used a Single Un-Partitioned Rectangle to Represent the John and Mary's Marbles.

Rectangles in a row group. The two remaining students, S9 and S10 drew rectangles in a row to represent the total number of marbles (the output). They then partitioned the left end of the rectangle and used this to represent the difference. From here S9 and S10's representations diverged. If the difference was small (under 10), S9 used dotted lines to mark in the individual rectangles to indicate the exact value of the difference. As shown in the left panel of Figure 7, when the difference and the inputs were big, lines were drawn into the rectangles but these did not represent the exact

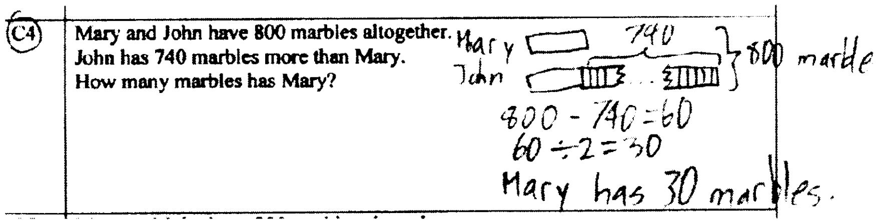


Figure 6. AT8 However When the Difference was Big (above 10), a Partly Partitioned Left and Right End of the Difference Rectangle with Dots in Between to Suggest That the Number of Rectangles Continue was Used to Represent the Difference.

number of marbles. Rather they were used to suggest continuity and hence large numbers. The right panel shows how S10 used un-partitioned rectangles in a single row to represent both the difference and the inputs.

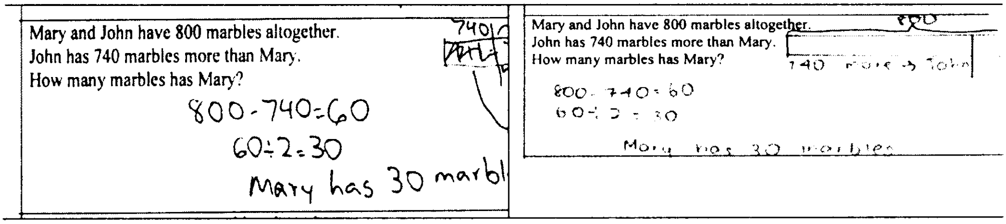


Figure 7. Left Panel Shows S9's Solution Where the Lines Need not Represent Actual Numbers. Right Panel, S10 Used Un-Partitioned Rectangles to Represent Both the Difference And the Inputs.

Eight students (AT1 – AT7, S10) who used un-partitioned rectangles could be described as potential abstract thinkers since they used almost similar sized rectangles for inputs ranging from 10 – 10 000. In fact the rectangles used in modeling items in sets C and D were shorter that those used in sets A and B. Two students, A8 and S9 were categorized as concrete-abstract thinkers. These students made use of the rectangles in two different ways: (i) to represent specific numbers and (ii) as abstract objects representing any number. This does not mean that abstract thinkers are more advanced in their thinking about the rectangles. What it does mean is that the concrete-abstract thinkers were flexible in the way they perceived the rectangles – as exact representation of specific numbers and representation for any number, a variable.

The interviewed data confirmed the inference based on students' written responses. The oral explanations showed that these students treated the rectangles as a variable where the lengths of the rectangles need not change to accommodate the change in the output. As long as the structure of the model was logical then the size of the rectangles did not matter. For example when asked "Suppose Mary and John have a total of one million marbles, can the same model be used?" all students agreed any of the model they had drawn for items B – D could be used as the size of the rectangles did not matter. The following students gave more in-depth explanations to support the case.

> AT1 explained that it was acceptable to use the same model since "the structure of the question is the same, just the numbers are different. It is the numbers that count but not the exact size of the boxes."

> AT2 emphasized that "So long as the person with the less marbles has a box which is smaller than the person that has more marbles, the model makes sense."

> AT8 who used dots to represent growing length explained that the dots showed the increasing number because "I use the dots to represent that there are many boxes. My primary 3 teacher taught me."

Students' explanations for the different models constructed for items across the different sections showed that students did not place great importance to the exact length of the rectangles. Most drew models of about the same size for different inputs. However there were models where the length of the rectangles was shorter for bigger inputs and longer for bigger inputs. This discrepancy when highlighted to individual students would draw a sheepish smile from them. However they provided very practical reasons for this discrepancy.

> The length of the rectangles does not matter as long as the numbers are in the right place and the drawing is logical. What happens if I did one problem today and I do the next problem on another day? I cannot remember how long was the rectangle I drew the day before. (Student AT1)

For the validation task, all the students chose the option that the size of the rectangles did not matter, thus agreeing that the solutions by the hypothetical students P and Q were correct. No further explanations were provided by students even though space was provided for them to record their justifications.

The interview data were more illuminating than the written responses in two ways. First students' explanations showed that they were prepared to be relative in their thinking in that it was not necessary that all model solutions should look alike. This was particularly true for the two students who used a single rectangle as a model. To them the structure of the comparison model was just as valid as their single rectangle model. Second all students agreed that as long as the structure of the model was logical, the length of the rectangles did not matter. If possible they would try and ensure that the lengths of the rectangles were proportional to one another. The general consensus was that the aim of drawing the model should be to help them solve problems as illustrated by the following reasons provided by some students.

> Does not matter how long. If it makes sense, makes it easier to solve the problem, makes you understand the problem. (S9.)

> Numbers are put in the right places. Rectangles are drawn correctly. If you understand the question you don't have to change the size of the rectangles. (AT2)

> When we do know the numbers, for example 80 and 60, then draw two rectangles the same, then add another box of 20. But if we don't know the number, you can draw any size. (AT3)

CONCLUSIONS

The written as well as the interview responses from students showed that the logical relationship among the rectangles was of paramount concern, not the length of the boxes. For these students the suggestion to use proportional reasoning to draw models was not cast in stone. While they treated the rectangles as receptacles of numbers they also knew that the same model drawing could be used to solve similarly structured questions. They all agreed that should they be asked to use model drawings to represent specific numbers, then the rectangles drawn would be proportional in length to one another. These students had constructed a new understanding of the role of the rectangles used in model drawing, knowledge not taught by teachers. From a pedagogical perspective, these students' knowledge should be used to facilitate their transition from arithmetic to letter-symbolic algebra.

At the beginning of this paper we raised the question of whether a more visual and pictorial representation for numbers would facilitate students work in letter-symbolic algebra. While this study does not answer that question directly, it nevertheless shows that students were not apprehensive when they had to treat and operate on rectangles as if they were variables. In fact they were receptive to the idea that other objects could be used to represent numbers. All the students replied with an affirmative yes to the final interview question: Can letters replace the role of rectangles? (See appendix D).

Curriculum planners and teachers may want to give these students' perception some thought as they try to ease primary students' transition to letter-symbolic algebra. That even primary school students could be potential abstract thinkers, may offer educators and policy makers a way forward to improve one particular but important aspect of algebra education, which is the notion of letters as variables. They may consider developing the notion of letters as variables through a concrete-pictorial approach and in stages and through arithmetic, a trajectory emphasized by MacLane and Birkhoff (op. cit) in the opening quote.

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Key words: Algebra, Children learning, Elementary school, Mathematics

APPENDIX A

List of Six Questions for Section D.

1. Mary and John have 8 000 marbles altogether.
   John has 6 000 marbles more than Mary.
   How many marbles has Mary?
2. Mary and John have 8 000 marbles altogether.
   John has 6 marbles more than Mary.
   How many marbles has Mary?
3. Mary and John have 8 000 marbles altogether.
   John has 60 marbles more than Mary.
   How many marbles has Mary?
4. Mary and John have 8 000 marbles altogether.
   John has 600 marbles more than Mary.
   How many marbles has Mary?
5. Mary and John have 9 000 marbles altogether.
   John has 3 000 marbles more than Mary.
   How many marbles has Mary?
6. Mary and John have 3 000 marbles altogether.
   John has 1 000 marbles more than Mary.
   How many marbles has Mary?

APPENDIX B

Questions That May Be Used in the Face-to-Face Semi-Structured Interview for Production Tasks

1. (a). What can you say about the method you used to solve these problems?
   (b). Can you use the models to solve Questions A to solve Questions D?
   Explain your answer.
2. Suppose Mary and John have a total of one million marbles, can the same model be used?
3. Let us look at question D2 and question D6. (For those who provided similar models)

(a) Do the models for these two questions look alike? Why is it possible to use the same model to solve these 2 questions?

(b) General question for those who think that the size of the boxes does not matter.
I thought in schools the teachers say you have to draw the boxes carefully, a longer box when the number is big and a shorter box when the number is small. But now you are saying that the size of the boxes does not matter. I am confused. Can you help me understand how you use the boxes in the model method?

4. Let us look at question D2 and question D6. (For those who applied proportional reasoning). The models for these two questions are different. For question D2, the difference between Mary and John is small, while in D6, the difference is very big. Why did you use different models to solve these two problems?

APPENDIX C

Questions That May Be Used in the Face-to-Face Semi-Structured Interview for Validation Tasks

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Type A questions

For Q1: I thought that Student P has drawn too big a box to represent 30 while student Q has drawn too small a box to represent 30. Please explain your answer.

For Q2: I thought that Student Q has drawn too big a box to represent 8. Please explain your answer.

For Q3: I thought that Student Q has drawn too small a box to represent what Mary and John have. Please explain your answer.

Type B questions

The following questions were used for all three type B questions.

Student P has used the same sized boxes to represent different numbers. If P is correct, why do you say P is correct?

Why is student Q correct when Student Q has used different sized boxes to represent different numbers?

APPENDIX D

Transfer of understanding

In the model method, rectangles are used to represent numbers.

(a) Can stars replace the role of rectangle?

(b) Can letters replace the role of rectangles?