Zero-free intervals of chromatic polynomials of hypergraphs

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Abstract

In this paper, we prove that $(-\infty, 0)$ is a zero-free interval for chromatic polynomials of a family $\mathcal{L}_0$ of hypergraphs and $(0,1)$ is a zero-free interval for chromatic polynomials of a subfamily $\mathcal{L}_0'$ of $\mathcal{L}_0$ of hypergraphs. These results extend known results on zero-free intervals of chromatic polynomials of graphs and hypergraphs.

MSC: 05C15, 05C31, 05C65

Keywords: graph; hypergraph; chromatic polynomial

1 Introduction

For any simple graph $G = (V, E)$ and any positive integer k, a *proper k-coloring of* G is a mapping $\phi : V \to \{1, 2, \ldots, k\}$ such that $\phi(u) \neq \phi(v)$ holds for each pair of adjacent vertices u and v in G. The *chromatic polynomial* $P(G, \lambda)$ of G is the function which counts the number of proper λ-colorings of G whenever λ is a positive integer. It was introduced by Birkhoff [6] in 1912 for planar graphs, in hope of proving the four-color conjecture, and extended to all graphs by Whitney [36] in 1932. Although Birkhoff's attempt failed, the study of chromatic polynomials is one of the active research areas

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in graph theory, especially the study of the zero distribution of chromatic polynomials [13, 14, 22, 24, 37, 38].

It was proved by Sokal [23] in 2004 that the zeros of chromatic polynomials of graphs are dense in the whole complex plane. It is also known that $(-\infty, 0), (0, 1)$ and $(1, 32/27]$ are the only three maximal zero-free real intervals for chromatic polynomials of graphs (see [13,24,31]), where the zero-free interval $(1, 32/27]$ was found by Jackson [13] and was proved to be the last such interval by Thomassen [24]. For certain subsets of graphs, Thomassen [25] showed that if G is any graph with a Hamiltonian path, then $P(G, \lambda)$ is zero-free in the interval $(1, 1.2955\cdots)$, while Dong and Jackson [9] proved that for any 3-connected planar graph G, $P(G, \lambda)$ is zero-free in the interval $(1, 1.2040\cdots)$.

The study of chromatic polynomials has been extended to chromatic polynomials of hypergraphs for many years. A *hypergraph* $\mathcal{H}$ is an ordered pair $(V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$, where $V(\mathcal{H})$ is a finite set, called the *vertex set* of $\mathcal{H}$, and $\mathcal{E}(\mathcal{H})$ is a set of non-empty subsets of $V(\mathcal{H})$, called the *edge set* of $\mathcal{H}$. Obviously, $\mathcal{E}(\mathcal{H}) \subseteq \{e \subseteq V(\mathcal{H}) : |e| \geq 1\}$. Thus a graph is a hypergraph $\mathcal{H}$ with $|e| \leq 2$ for each $e \in \mathcal{E}(\mathcal{H})$. Regarding colorings of hypergraphs, there are several different definitions, such as strong proper colorings [1] and $\mathcal{C}$-colorings [32]. In this paper, we work with weak proper colorings of hypergraphs which were introduced by Erdős and Hajnal [11] in 1966. For any integer $k \geq 1$, a *weak proper k-coloring* of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is an assignment of colors from $\{1, \ldots, k\}$ to the vertices so that each edge contains at least two vertices with different colors, i.e., there is a mapping $\phi : V \to \{1, \ldots, k\}$ such that $|\{\phi(v) : v \in e\}| \geq 2$ holds for each $e \in \mathcal{E}$. The *chromatic polynomial* of $\mathcal{H}$, denoted by $P(\mathcal{H}, \lambda)$, is the function which counts the number of weak proper k-colorings of $\mathcal{H}$ whenever $\lambda = k$ is a positive integer, and it is indeed a polynomial in λ [8, 26].

A weak proper k-coloring of a hypergraph $\mathcal{H}$ is actually a proper k-coloring of a graph whenever $\mathcal{H}$ is a graph. In such case, $P(\mathcal{H}, \lambda)$ is a chromatic polynomial of a graph. Thus chromatic polynomials of hypergraphs are generalizations of chromatic polynomials of graphs. Many properties of chromatic polynomials of graphs have been extended to chromatic polynomials of hypergraphs (for example, see [2–4, 7, 8, 26–30, 33–35]). Meanwhile in [39], the authors of this article also showed that some properties of chromatic polynomials of hypergraphs do not hold for chromatic polynomials of graphs. One of these properties is that chromatic polynomials of hypergraphs have real zeros that are dense in the whole set of real numbers, while chromatic polynomials of graphs have three zero-free intervals. It is natural for people to look for a suitable way of extending the result that $(-\infty, 0)$ and $(0, 1)$ are zero-free intervals of chromatic polynomials of graphs to chromatic polynomials of hypergraphs.

A *cycle* C in $\mathcal{H}$ is defined to be a sequence of alternating vertices and edges: $(v_1, e_1, v_2, e_2, \ldots, v_t, e_t, v_1)$, where $t \geq 2$, $v_1, \ldots, v_t$ are pairwise distinct vertices and $e_1, \ldots, e_t$ are pairwise distinct edges, such that $\{v_i, v_{i+1}\} \subseteq e_i$ for $i \in \{1, \ldots, t\}$, where $v_{t+1} = v_1$. Sometimes, a cycle C is also written as $(e_1, \ldots, e_t)$ for short. In this case, we assume that there exist pairwise distinct vertices $v_1, \ldots, v_t$ such that $v_{i+1} \in e_i \cap e_{i+1}$ for all $i = 0, 1, \ldots, t-1$, where $e_0 = e_t$.

Let $\mathcal{L}_1$ be the set of hypergraphs in which each edge has an even size and each cycle $C = (e_1, \ldots, e_r)$ contains e_i with $|e_i| = 2$ for some $i \in \{1, \ldots, r\}$. Dohmen [8] proved that for any hypergraph $\mathcal{H}$ in $\mathcal{L}_1$, $P(\mathcal{H}, \lambda) \neq 0$ holds for any real $\lambda \in (-\infty, 0)$.

Theorem 1.1 ([8]) *If* $\mathcal{H} = (V, \mathcal{E})$ *is a hypergraph in* $\mathcal{L}_1$*, then* $(-1)^{|V|} P(\mathcal{H}, \lambda) > 0$ *holds for all real* $\lambda \in (-\infty, 0)$*.*

Theorem 1.1 certainly includes the fact that chromatic polynomials of graphs has no real zero in the interval $(-\infty, 0)$. As we know, this result is currently the only known result on zero-free intervals for chromatic polynomials of hypergraphs.

Observe that the hypergraph $\mathcal{H}$ in Figure 1 does not belong to $\mathcal{L}_1$, as there is a cycle (e_1, e_2, e_3) in $\mathcal{H}$ with $|e_i| > 2$ for all $i = 1, 2, 3$. It can be calculated that

$$P(\mathcal{H}, \lambda) = \lambda^6 (\lambda - 1)^4 (\lambda^3 - 2)(\lambda^2 + \lambda + 1)^2,$$

which shows that although $\mathcal{H}$ is not in $\mathcal{L}_1$, its chromatic polynomial $P(\mathcal{H}, \lambda)$ is zero-free in the interval $(-\infty, 0)$.

In this paper, we first extend Dohmen's result to a larger set of hypergraphs, which definitely includes the hypergraph shown in Figure 1. Further we prove the existence of an infinite set of hypergraphs with at least one edge of size greater than 2 whose chromatic polynomials are zero-free in the interval $(0, 1)$.

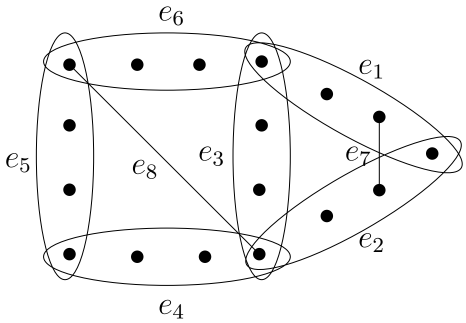

Figure 1: A hypergraph in $\mathcal{L}_0 \setminus \mathcal{L}_1$

Let $\mathcal{L}_0$ be the set of hypergraphs $\mathcal{H}$ in which each edge has an even size and each cycle $C = (e_1, \dots, e_r)$ contains distinct vertices u and v in the set $\bigcup_{1\le i\le r} e_i$ forming an edge in $\mathcal{H}$. For instance, the hypergraph $\mathcal{H}$ shown in Figure 1 is in $\mathcal{L}_0 \setminus \mathcal{L}_1$, which also implies that $\mathcal{L}_1$ is a proper subset of $\mathcal{L}_0$. Actually there are infinitely many hypergraphs in $\mathcal{L}_0 \setminus \mathcal{L}_1$. Let $\mathcal{L}'_0$ be the set of hypergraphs $\mathcal{H} = (V, \mathcal{E}) \in \mathcal{L}_0$ containing a connected spanning subhypergraph $(V, \mathcal{E}_2)$, where $\mathcal{E}_2 = \{e \in \mathcal{E} : |e| = 2\}$. Clearly, $\mathcal{L}'_0 \subset \mathcal{L}_0$.

We prove the following two results on the chromatic polynomials of hypergraphs in $\mathcal{L}_0$ and $\mathcal{L}'_0$ in this article.

Theorem 1.2 *If $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph in $\mathcal{L}_0$, then $(-1)^{|V|}P(\mathcal{H}, \lambda) > 0$ holds for all real $\lambda \in (-\infty, 0)$.*

Theorem 1.2 implies Theorem 1.1 directly, as $\mathcal{L}_1 \subset \mathcal{L}_0$.

Theorem 1.3 *If $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph in $\mathcal{L}'_0$, then*

(i) $(-1)^{|V|+1}P(\mathcal{H}, \lambda) > 0$ *holds for all real* $\lambda \in (0, 1)$*; and*

(ii) $P(\mathcal{H}, \lambda)$ *has no multiple zero at* $\lambda = 0$.

Note that the conclusion of Theorem 1.3 fails for some hypergraphs in $\mathcal{L}_0 \setminus \mathcal{L}'_0$. For instance, the hypergraph $\mathcal{H}$ in Figure 2 belongs to the set $\mathcal{L}_0 \setminus \mathcal{L}'_0$ and its chromatic polynomial is

$$P(\mathcal{H}, \lambda) = \lambda(\lambda - 1)^2(\lambda - 2)(\lambda^2 - 3\lambda + 1),$$

which has a real zero at around 0.38.

The proofs of Theorems 1.2 and 1.3 are given in Section 4.

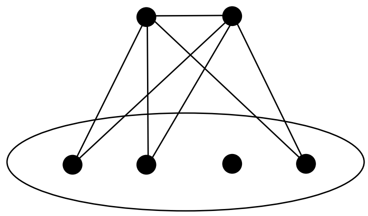

Figure 2: A hypergraph in $\mathcal{L}_0 \setminus \mathcal{L}'_0$

2 Preliminaries

In this section, we present several known results on chromatic polynomials of hypergraphs, which will be applied later. The first one follows directly from the definition of weak proper colorings of a hypergraph.

Proposition 2.1 [17] *Let e_1 and e_2 be two edges in a hypergraph $\mathcal{H}$. If $e_1 \subseteq e_2$, then*

$$P(\mathcal{H}, \lambda) = P(\mathcal{H} - e_2, \lambda),$$

where $\mathcal{H} - e_2$ is the hypergraph obtained from $\mathcal{H}$ by removing e_2.

Proposition 2.1 shows that we need only to consider Sperner hypergraphs when studying chromatic polynomials of hypergraphs, where a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is called *Sperner* if $e_1 \not\subseteq e_2$ holds for any distinct edges $e_1, e_2 \in \mathcal{E}$.

For any hypergraph $\mathcal{H} = (V, \mathcal{E})$, its *Sperner subhypergraph* is defined to be $\mathcal{H}^* = (V, \mathcal{E}^*)$, where $\mathcal{E}^*$ is the minimal subset of $\mathcal{E}$ such that for each $e \in \mathcal{E} \setminus \mathcal{E}^*$, there exists $e' \in \mathcal{E}^*$ with $e' \subseteq e$. That means, $\mathcal{H}^*$ can be obtained from $\mathcal{H}$ by removing any edge $e \in \mathcal{E}$ whenever $e' \subseteq e$ holds for another edge $e' \in \mathcal{E}$. It is not difficult to show that $\mathcal{E}^*$ is uniquely determined by $\mathcal{E}$. Obviously, $\mathcal{H}^*$ is Sperner, and $\mathcal{H}^* = \mathcal{H}$ when $\mathcal{H}$ is Sperner. Moreover, by Proposition 2.1,

$$P(\mathcal{H}, \lambda) = P(\mathcal{H}^*, \lambda). \tag{1}$$

The following proposition can be obtained from the definition of Sperner hypergaphs directly.

Proposition 2.2 *If $\mathcal{H} = (V, \mathcal{E})$ is Sperner and e_0 is an edge in $\mathcal{H}$ with $|e_0| = 2$, then $|e_0 \cap e| \leq 1$ holds for any edge $e \in \mathcal{E} \setminus \{e_0\}$.*

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and $V_0 \subset V$. Let $\mathcal{H} \cdot V_0$ denote the hypergraph obtained from $\mathcal{H}$ by identifying all vertices in V_0 as one, i.e., $\mathcal{H} \cdot V_0$ is the hypergraph with vertex set $(V - V_0) \cup \{w\}$ and edge set

$$\{e \in \mathcal{E} : e \cap V_0 = \emptyset\} \cup \{(e - V_0) \cup \{w\} : e \in \mathcal{E}, e \cap V_0 \neq \emptyset\},$$

where w is the new vertex produced when $\mathcal{H} \cdot V_0$ is obtained from $\mathcal{H}$ by identifying all vertices in V_0. For an edge e in $\mathcal{H}$, let $\mathcal{H}/e$ denote the hypergraph $(\mathcal{H} - e) \cdot e$. We also say that $\mathcal{H}/e$ is obtained from $\mathcal{H}$ by *contracting* the edge e.

The deletion-contraction formula for chromatic polynomials of graphs is very important for the computation of this polynomial [5,6,10,20,21]. It was extended to chromatic polynomials of hypergraphs by Jones [16].

Theorem 2.3 [16] *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. For any $e \in \mathcal{E}$,*

$$P(\mathcal{H}, \lambda) = P(\mathcal{H} - e, \lambda) - P(\mathcal{H}/e, \lambda). \tag{2}$$

3 $\mathcal{L}_0$ is deletion and contraction closed

In this section, we show that for any Sperner hypergraph $\mathcal{H} \in \mathcal{L}_0$ with at least one edge, there is an edge e_0 in $\mathcal{H}$ such that both $\mathcal{H} - e_0$ and $(\mathcal{H}/e_0)^*$ belong to $\mathcal{L}_0$.

For any hypergraph $\mathcal{H} = (V, \mathcal{E})$ and $e \in \mathcal{E}$, let $\mathcal{H}/e$ be written as $(V/e, \mathcal{E}/e)$. By the notation for the Sperner subhypergraph, $(\mathcal{H}/e)^*$ is written as $(V/e, (\mathcal{E}/e)^*)$.

Lemma 3.1 *Let $\mathcal{H} = (V, \mathcal{E}) \in \mathcal{L}_0$ and $e_0 \in \mathcal{E}$. Assume that $\mathcal{H}$ is Sperner. Then,*

$$(\mathcal{E}/e_0)^* \subseteq \{e \in \mathcal{E} : e \cap e_0 = \emptyset\} \cup \{(e - e_0) \cup \{w\} : |e \cap e_0| = 1, e \in \mathcal{E}\}, \tag{3}$$

where $w \notin V$.

Proof. By the definition of $\mathcal{H}/e_0$, we have

$$\mathcal{E}/e_0 = \{e \in \mathcal{E} : e \cap e_0 = \emptyset\} \cup \{(e - e_0) \cup \{w\} : e \in \mathcal{E}, e \cap e_0 \neq \emptyset\}. \tag{4}$$

By the definition of $(\mathcal{H}/e_0)^*$ and (4), to prove (3), it suffices to show that for any $e_1 \in \mathcal{E}$ with $|e_1 \cap e_0| \geq 2$, there exists $e_2 \in \mathcal{E}$ with $|e_2 \cap e_0| = 1$ such that $e_2 - e_0 \subseteq e_1 - e_0$.

Let $e_1 \in \mathcal{E}$ with $\{u, v\} \subseteq e_1 \cap e_0$, as shown in Figure 3. Observe that (u, e_0, v, e_1, u) forms a cycle in $\mathcal{H}$. Since $\mathcal{H} \in \mathcal{L}_0$, there exists an edge $e_2 = \{x, y\} \in \mathcal{E}$ such that $e_2 \subseteq e_0 \cup e_1$. Moreover, $e_2 \not\subseteq e_i$ for $i \in \{0, 1\}$ as $\mathcal{H}$ is Sperner. Without loss of generality, we assume that $x \in e_1 - e_0$ and $y \in e_0 - e_1$, as shown in Figure 3. Clearly, $e_2 - e_0 = \{x\} \subseteq e_1 - e_0$, as required. Thus the result holds. □

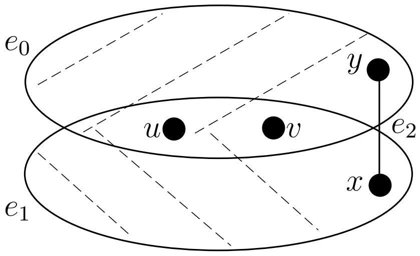

Figure 3: A hypergraph $\mathcal{H} \in \mathcal{L}_0$ with $|e_1 \cap e_0| \geq 2$

In the following, we show that for any Sperner hypergraph $\mathcal{H} = (V, \mathcal{E})$ in $\mathcal{L}_0$ with $\mathcal{E} \neq \emptyset$, there always exists some edge e_0 such that $\mathcal{H} - e_0 \in \mathcal{L}_0$ and $(\mathcal{H}/e_0)^* \in \mathcal{L}_0$.

Lemma 3.2 *Let $\mathcal{H} = (V, \mathcal{E}) \in \mathcal{L}_0$ and e_0 be the edge of $\mathcal{H}$ with the largest size. If $\mathcal{H}$ is Sperner, then $\mathcal{H} - e_0 \in \mathcal{L}_0$ and $(\mathcal{H}/e_0)^* \in \mathcal{L}_0$.*

Proof. If $|e_0| = 2$, then $|e| = 2$ for every $e \in \mathcal{E}$, and so $\mathcal{H}$ is a graph in $\mathcal{L}_0$. Obviously, $\mathcal{H} - e_0 \in \mathcal{L}_0$ and $(\mathcal{H}/e_0)^* \in \mathcal{L}_0$.

Now assume that $|e_0| > 2$. Clearly, $\mathcal{H} - e_0 \in \mathcal{L}_0$ by the definition of $\mathcal{L}_0$. In what follows, we show that $(\mathcal{H}/e_0)^* \in \mathcal{L}_0$.

Let $(\mathcal{H}/e_0)^*$ be the hypergraph $(V/e_0, (\mathcal{E}/e_0)^*)$. For any vertex u in $(\mathcal{H}/e_0)^*$, by the definition of contracting, either $u \in V - e_0$ or $u = w$, where w is the new vertex in $\mathcal{H}/e_0$ after contracting e_0 in $\mathcal{H}$. For any edge $e^* \in (\mathcal{E}/e_0)^*$, by Lemma 3.1, either $e^* \in \{e \in \mathcal{E} : e \cap e_0 = \emptyset\}$ or $e^* \in \{(e - e_0) \cup \{w\} : e \in \mathcal{E}, |e \cap e_0| = 1\}$, implying that e^* always has the even size as $\mathcal{H} \in \mathcal{L}_0$. Thus, to show that $(\mathcal{H}/e_0)^* \in \mathcal{L}_0$, it remains proving the following claim.

Claim: For any cycle $C^* = (u_1, e_1^*, u_2, e_2^*, \ldots, u_t, e_t^*, u_1)$ in $(\mathcal{H}/e_0)^*$, there exist $u, v \in \bigcup_{1\le i\le t} e_i^*$ such that $\{u, v\} \in (\mathcal{E}/e_0)^*$.

To prove this claim, it suffices to show that the claim holds for each minimal cycle C^* in $(\mathcal{H}/e_0)^*$, i.e., a cycle that does not contain any other cycle in $(\mathcal{H}/e_0)^*$. In the following, we assume that C^* is a minimal cycle in $(\mathcal{H}/e_0)^*$. It is trivial if $|e_i^*| = 2$ for some $i \in \{1, \ldots, t\}$. Thus we also assume that $|e_i^*| > 2$ for all $i = 1, \ldots, t$.

By Lemma 3.1, there exist edges $e_1, \ldots, e_t$ in $\mathcal{H}$ with the following properties: for all $i = 1, \ldots, t$,

(a) $|e_i \cap e_0| \le 1$;

(b) $e_i^* = e_i$ when $e_i \cap e_0 = \emptyset$; and $e_i^* = (e_i - e_0) \cup \{w\}$ otherwise.

Let $P_0 = e_0 \cap \bigcup_{1\le i\le t} e_i$. Clearly, $\bigcup_{1\le i\le t} e_i^* = \bigcup_{1\le i\le t} e_i$ when $P_0 = \emptyset$, and $\bigcup_{1\le i\le t} e_i^* = (\bigcup_{1\le i\le t} e_i - P_0) \cup \{w\}$ otherwise.

We are now going to prove the claim on a case-by-case basis.

Case 1: $w \notin \{u_1, \ldots, u_t\}$.

In this case, either $w \notin \bigcup_{1\le i\le t} e_i^*$ or $w \in \bigcup_{1\le i\le t} e_i^*$ but $w \notin \{u_1, \ldots, u_t\}$. Thus, there is a cycle $C = (u_1, e_1, u_2, e_2, \ldots, u_t, e_t, u_1)$ in $\mathcal{H}$. Since $\mathcal{H} \in \mathcal{L}_0$, there is an edge $e' = \{u, v\} \in \mathcal{E}$ such that $e' \subseteq \bigcup_{1\le i\le t} e_i$. By Proposition 2.2, $|e' \cap e_0| \le 1$.

If $e' \cap e_0 = \emptyset$, then $e' \in \mathcal{E}/e_0$ by the definition of $\mathcal{H}/e_0$. Further, $e' \in (\mathcal{E}/e_0)^*$ as $|e'| = 2$. Since $e' \subseteq \bigcup_{1\le i\le t} e_i$ and $e' \cap e_0 = \emptyset$, we have that $e' \cap P_0 = \emptyset$, implying that $e' \subseteq \bigcup_{1\le i\le t} e_i^*$.

If $|e' \cap e_0| = 1$, say $e' \cap e_0 = \{u\}$, then $\{v, w\} \in \mathcal{E}/e_0$ by the definition of $\mathcal{H}/e_0$. Thus $\{v, w\} \in (\mathcal{E}/e_0)^*$. Since $e' \subseteq \bigcup_{1\le i\le t} e_i$ and $e' \cap e_0 = \{u\}$, we have that $P_0 \ne \emptyset$ which implies $\bigcup_{1\le i\le t} e_i^* = (\bigcup_{1\le i\le t} e_i - P_0) \cup \{w\}$. Thus $\{v, w\} \subseteq \bigcup_{1\le i\le t} e_i^*$.

Hence the claim holds in this case.

Case 2: $w \in \{u_1, \ldots, u_t\}$.

Without loss of generality, suppose that $w = u_1$. Since $w = u_1 \in e_1^* \cap e_t^*$, by Lemma 3.1, we have $|e_1 \cap e_0| = |e_t \cap e_0| = 1$. By assumption, C^* is a minimal cycle in $(\mathcal{H}/e_0)^*$, implying that $w \notin e_i^*$ for all $i = 2, \ldots, t-1$, and so $e_i \cap e_0 = \emptyset$ for all $i = 2, 3, \ldots, t-1$ by Lemma 3.1.

Case 2.1: $e_1 \cap e_0 = e_t \cap e_0 = \{x\}$ for some $x \in V$.

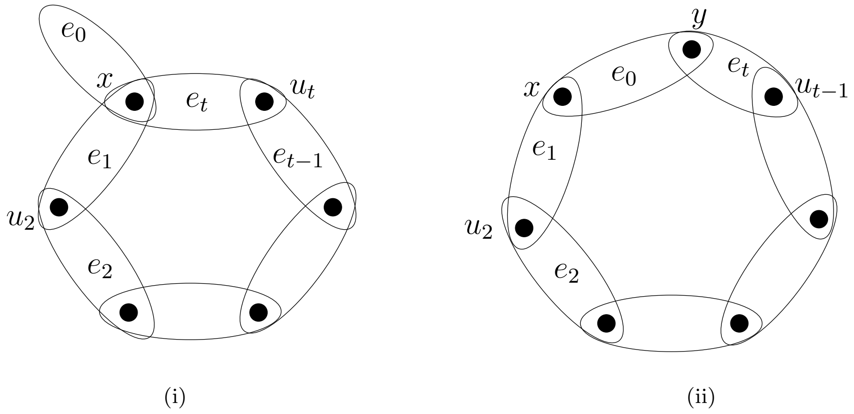

Figure 4: A cycle of $\mathcal{H}$ in Cases 2.1 and 2.2

In this subcase, $e_1 = (e_1^* \setminus \{w\}) \cup \{x\}$, $e_t = (e_t^* \setminus \{w\}) \cup \{x\}$ and $e_i = e_i^*$ for all $i = 2, 3, \cdots, t-1$. Thus $C = (x, e_1, u_2, e_2, u_3, \ldots, u_t, e_t, x)$ is a cycle in $\mathcal{H}$, as shown in Figure 4 (i). Since $\mathcal{H} \in \mathcal{L}_0$, there is an edge $e' = \{u, v\} \in \mathcal{E}$ such that $e' \subseteq \bigcup_{1 \le i \le t} e_i$.

If $x \notin \{u, v\}$, then $e' \in (\mathcal{E}/e_0)^*$ and $e' \subseteq \bigcup_{1 \le i \le t} e_i^*$.

If $x \in \{u, v\}$, say $x = u$, then $\{w, v\} \in (\mathcal{E}/e_0)^*$ and $\{w, v\} \subseteq \bigcup_{1 \le i \le t} e_i^*$.

Thus the claim holds for this subcase.

Case 2.2: $e_1 \cap e_0 = \{x\}$ and $e_t \cap e_0 = \{y\}$, where $x, y \in V$ and $x \neq y$.

In this subcase, $e_1 = (e_1^* \setminus \{w\}) \cup \{x\}$, $e_t = (e_t^* \setminus \{w\}) \cup \{y\}$ and $e_i = e_i^*$ for all $i = 2, 3, \cdots, t-1$. Thus $C = \{y, e_0, x, e_1, u_2, e_2, u_3, e_3, \ldots, u_t, e_t, y\}$ is a cycle in $\mathcal{H}$, as shown in Figure 4 (ii). Since $\mathcal{H} \in \mathcal{L}_0$, there is an edge $e' = \{u, v\} \in \mathcal{E}$ such that $e' \subseteq \bigcup_{0 \le i \le t} e_i$. By Proposition 2.2, $|e' \cap e_0| \le 1$.

If $e' \cap e_0 = \emptyset$, then $e' \in \mathcal{E}/e_0$ by the definition of $\mathcal{H}/e_0$, implying that $e' \in (\mathcal{E}/e_0)^*$ as $|e'| = 2$. Since $e' \cap e_0 = \emptyset$ and $\{x, y\} \subseteq e_0$, we have that $e' \subseteq \left(\bigcup_{1 \le i \le t} e_i\right) \setminus \{x, y\}$, implying that $e' \subseteq \bigcup_{1 \le i \le t} e_i^*$.

If $|e' \cap e_0| = 1$, say $e' \cap e_0 = \{u\}$, then $\{w, v\} \in \mathcal{E}/e_0$ by the definition of $\mathcal{H}/e_0$, which also implies that $\{w, v\} \in (\mathcal{E}/e_0)^*$.

As $v \notin e_0$, we have that $v \in \left(\bigcup_{1 \le i \le t} e_i\right) \setminus \{x, y\}$, implying that $\{w, v\} \in \bigcup_{1 \le i \le t} e_i^*$.

Thus, the claim holds for this subcase.

Therefore, the claim holds in Case 2 and the proof is complete. □

4 Proving the main results

Now we are ready to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Let n be the number of vertices in $\mathcal{H}$ and m be the number of edges in $\mathcal{H}$. We shall prove the statement by induction on m.

When $m = 0$, $\mathcal{H}$ is an empty graph and $P(\mathcal{H}, \lambda) = \lambda^n$. It is obvious that $(-1)^n P(\mathcal{H}, \lambda) > 0$ for any $\lambda \in (-\infty, 0)$.

Assume that Theorem 1.2 holds for all hypergraphs in $\mathcal{L}_0$ with less than m ($m \geq 1$) edges. Now let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph in $\mathcal{L}_0$, where $|V| = n$ and $|\mathcal{E}| = m$. By (1), we can assume that $\mathcal{H}$ is Sperner.

Let e_0 be the edge in $\mathcal{H}$ with the largest size. If $|e_0| = 2$, then $\mathcal{H}$ is a graph and Theorem 1.2 has been proved that it holds for a graph (see [10, 13, 20, 21]).

If $|e_0| > 2$, by Theorem 2.3 and (1), we have that

$$P(\mathcal{H}, \lambda) = P(\mathcal{H} - e_0, \lambda) - P(\mathcal{H}/e_0, \lambda) = P(\mathcal{H} - e_0, \lambda) - P((\mathcal{H}/e_0)^*, \lambda). \quad (5)$$

As $\mathcal{H}$ is a Sperner hypergraph in $\mathcal{L}_0$, by Lemma 3.2, both $\mathcal{H} - e_0$ and $(\mathcal{H}/e_0)^*$ belong to $\mathcal{L}_0$. Thus, by the inductive assumption, for any real $\lambda \in (-\infty, 0)$,

$$(-1)^n P(\mathcal{H} - e_0, \lambda) > 0 \quad \text{and} \quad (-1)^{n-|e_0|+1} P((\mathcal{H}/e_0)^*, \lambda) > 0. \quad (6)$$

As $\mathcal{H} \in \mathcal{L}_0$, $|e_0|$ is even. Thus, from (5) and (6), for any real $\lambda \in (-\infty, 0)$,

$$\begin{aligned}(-1)^n P(\mathcal{H}, \lambda) =& (-1)^n (P(\mathcal{H} - e_0, \lambda) - P((\mathcal{H}/e_0)^*, \lambda)) \\ =& \underbrace{(-1)^n P(\mathcal{H} - e_0, \lambda)}_{>0} + \underbrace{(-1)^{|e_0|}}_{>0} \underbrace{(-1)^{n-|e_0|+1} P((\mathcal{H}/e_0)^*, \lambda)}_{>0} \\ >& 0.\end{aligned}$$

The proof is complete. □

For any hypergraph $\mathcal{H}$, let

$$Q(\mathcal{H}, \lambda) = \frac{1}{\lambda} \cdot P(\mathcal{H}, \lambda). \quad (7)$$

Clearly, $Q(\mathcal{H}, \lambda)$ is also a polynomial in λ, as $P(\mathcal{H}, \lambda)$ is a polynomial in λ and $P(\mathcal{H}, 0) = 0$ (see [26]). To prove Theorem 1.3, it suffices to establish the result below.

Theorem 4.1 *If $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph in $\mathcal{L}'_0$, then for any real $\lambda \in [0, 1)$,*

$$(-1)^{|V|+1} Q(\mathcal{H}, \lambda) > 0. \tag{8}$$

Proof. Let $\mathcal{E}'(\mathcal{H}) = \{e \in \mathcal{E} : |e| > 2\}$ and $m' = |\mathcal{E}'(\mathcal{H})|$. We shall prove the statement by induction on m'. When $m' = 0$, $\mathcal{H} \in \mathcal{L}'_0$ implies that $\mathcal{H}$ is a connected graph. Thus (8) holds in this case (see [10, 13]).

Assume that (8) holds for all hypergraphs $\mathcal{H}$ in $\mathcal{L}'_0$ with $|\mathcal{E}'(\mathcal{H})| < m'$ ($m' \geq 1$).

Now let $\mathcal{H} = (V, \mathcal{E})$ be a Sperner hypergraph in $\mathcal{L}'_0$ with $|\mathcal{E}'(\mathcal{H})| = m'$. Let $e_0 \in \mathcal{E}'(\mathcal{H})$. By Theorem 2.3 and (1),

$$P(\mathcal{H}, \lambda) = P(\mathcal{H} - e_0, \lambda) - P(\mathcal{H}/e_0, \lambda) = P(\mathcal{H} - e_0, \lambda) - P((\mathcal{H}/e_0)^*, \lambda). \tag{9}$$

By (7) and (9),

$$Q(\mathcal{H}, \lambda) = Q(\mathcal{H} - e_0, \lambda) - Q((\mathcal{H}/e_0)^*, \lambda). \tag{10}$$

As $|e_0| > 2$ and $\mathcal{H} \in \mathcal{L}'_0$, $\mathcal{H} - e_0 \in \mathcal{L}'_0$ holds.

By Lemma 3.2, $(\mathcal{H}/e_0)^* \in \mathcal{L}_0$. Let $\mathcal{H}^2$ be the hypergraph $(V, \mathcal{E}_2)$. As $\mathcal{H} \in \mathcal{L}'_0$, $\mathcal{H}^2$ is connected. By Proposition 2.2, $|e' \cap e_0| \leq 1$ for each $e' \in \mathcal{E}_2$. Thus $\mathcal{H}^2/e_0$ is connected, implying that $(\mathcal{H}/e_0)^* \in \mathcal{L}'_0$.

Since $|e_0| > 2$, $|\mathcal{E}'(\mathcal{H} - e_0)| < m'$ and $|\mathcal{E}'((\mathcal{H}/e_0)^*)| < m'$. Thus, by inductive assumption, for any real $\lambda \in [0, 1)$,

$$(-1)^{|V|+1} Q(\mathcal{H} - e_0, \lambda) > 0 \quad \text{and} \quad (-1)^{|V|-|e_0|+2} Q((\mathcal{H}/e_0)^*, \lambda) > 0. \tag{11}$$

As $\mathcal{H} \in \mathcal{L}_0$, $|e_0|$ is even. Thus by (10) and (11), for any real $\lambda \in [0, 1)$,

$$\begin{aligned}(-1)^{|V|+1} Q(\mathcal{H}, \lambda) =& (-1)^{|V|+1}(Q(\mathcal{H} - e_0, \lambda) - Q((\mathcal{H}/e_0)^*, \lambda)) \\ =& \underbrace{(-1)^{|V|+1} Q(\mathcal{H} - e_0, \lambda)}_{>0} + \underbrace{(-1)^{|e_0|}}_{>0} \underbrace{(-1)^{|V|-|e_0|+2} Q((\mathcal{H}/e_0)^*, \lambda)}_{>0} \\ >& 0.\end{aligned}$$

The proof is complete. □

5 Further study

We are not sure if Theorems 1.2 and 1.3 can be extended to larger families of hypergraphs. Especially for Theorem 1.3, we are not sure if its conclusion holds for some connected hypergraph $\mathcal{H} = (V, \mathcal{E})$ whose spanning subhypergraph $(V, \mathcal{E}_2)$ is not connected.

Problem 5.1 *Find a sufficient condition weaker than that in Theorem 1.3 for a hypergraph* $\mathcal{H} = (V, \mathcal{E})$ *such that* $P(\mathcal{H}, \lambda) \neq 0$ *for all* $\lambda \in (0, 1)$.

We wonder if Theorems 1.2 and 1.3 can be extended to the chromatic polynomials of mixed hypergraphs. A *mixed hypergraph* is a triple $\mathcal{H} = (V, \mathcal{C}, \mathcal{D})$, where V is a finite set, called the *vertex set* of $\mathcal{H}$, and $\mathcal{C}$ and $\mathcal{D}$ are collections of subsets of V, called the $\mathcal{C}$*-edge set* and $\mathcal{D}$*-edge set* respectively. A *proper* k*-coloring* of $\mathcal{H} = (V, \mathcal{C}, \mathcal{D})$ is a mapping $\phi : V \to \{1, 2, \ldots, k\}$ such that $|\{\phi(v) : v \in e\}| < |e|$ holds for each $e \in \mathcal{C}$ and $|\{\phi(v) : v \in e\}| \geq 2$ holds for each $e \in \mathcal{D}$. The coloring of mixed hypergraphs was introduced by Voloshin [33] in 1995. Since then, the study of colorings of mixed hypergraphs has developed into a separate topic and it has been applied in many fields, such as database management, channel assignments and cyber security [12, 15, 18, 19].

The *chromatic polynomial* of a mixed hypergraph $\mathcal{H} = (V, \mathcal{C}, \mathcal{D})$ is the function, denoted by $P_{mix}(\mathcal{H}, \lambda)$, which counts the number of proper λ-colorings of $\mathcal{H}$ whenever λ is a positive integer. Evidently, a weak proper k-coloring of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is actually a proper k-coloring of the mixed hypergraph $\mathcal{H}' = (V, \emptyset, \mathcal{E})$, implying that $P(\mathcal{H}, \lambda) = P_{mix}(\mathcal{H}', \lambda)$. Thus, the chromatic polynomials of mixed hypergraphs extend the chromatic polynomials of hypergraphs.

Problem 5.2 *Extend Theorems 1.2 and 1.3 to mixed hypergraphs.*

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References

[1] G. Agnarsson and M. M. Halldórsson. Strong colorings of hypergraphs. In *International Workshop on Approximation and Online Algorithms*, pages 253–266. Springer, 2004.

[2] J. A. Allagan. The chromatic polynomials of some linear uniform hypergraphs. *Congr. Numer.*, **187**:156–160, 2007.

[3] J. A. Allagan. Chromatic polynomials of some (m, l)-hyperwheels. *Comput. Sci. J. Moldova*, **22**:21–36, 2014.

[4] J. A. Allagan and D. Slutzky. Chromatic polynomials of some mixed hypergraphs. *Australas. J. Combin.*, **58**:197–213, 2014.

[5] N. L. Biggs. *Algebraic Graph Theory*. Cambridge University Press, 1993.

[6] G. D. Birkhoff and D. C. Lewis. Chromatic polynomials. *Trans. Amer. Math. Soc.*, **60**:355–451, 1946.

[7] M. Borowiecki and E. Łazuka. Chromatic polynomials of hypergraphs. *Discuss. Math. Graph Theory*, **20**:293–301, 2000.

[8] K. Dohmen. A broken-circuits-theorem for hypergraphs. *Arch. Math.*, **64**:159–162, 1995.

[9] F. Dong and B. Jackson. A zero-free interval for chromatic polynomials of nearly 3-connected plane graphs. *SIAM J. Discrete Math.*, **25**:1103–1118, 2011.

[10] F. Dong, K. M. Koh, and K. L. Teo. *Chromatic Polynomials and Chromaticity of Graphs*. World Scientific, 2005.

[11] P. Erdős and A. Hajnal. On chromatic number of graphs and set-systems. *Acta Math. Hungar.*, **17**:61–99, 1966.

[12] K. Hirotsugu and M. Tetsuya. Access control model for the inference attacks with access histories. In *2017 IEEE 41st Annual Computer Software and Applications Conference (COMPSAC)*, pages 498–503. IEEE, 2017.

[13] B. Jackson. A zero-free interval for chromatic polynomials of graphs. *Combin. Probab. Comput.*, **2**:325–336, 1993.

[14] B. Jackson. Zeros of chromatic and flow polynomials of graphs. *J. Geom.*, **76**:95–109, 2003.

[15] A. Jaffe, T. Moscibroda, and S. Sen. On the price of equivocation in byzantine agreement. In *Proceedings of the 2012 ACM Symposium on Principles of Distributed Computing*, pages 309–318. ACM, 2012.

[16] R. P. Jones. *Colourings of hypergraphs.* PhD thesis, University of London, 1976.

[17] R. P. Jones. Some results of chromatic hypergraph theory proved by "reduction to graphs". In *Colloque CNRS, ProblmesCombinatories et Thorie des Graphes*, volume **260**, pages 249–250, 1976.

[18] D. Král'. Mixed hypergraphs and other coloring problems. *Discrete Math.*, **307**:923–938, 2007.

[19] C. Li, M. Hurfin, Y. Wang, and L. Yu. Towards a restrained use of non-equivocation for achieving iterative approximate byzantine consensus. In *2016 IEEE International Parallel and Distributed Processing Symposium (IPDPS)*, pages 710–719. IEEE, 2016.

[20] R. C. Read. An introduction to chromatic polynomials. *J. Combin. Theory*, **4**:52–71, 1968.

[21] R. C. Read and W. T. Tutte. Chromatic polynomials. In *Selected Topics in Graph Theory 3*, pages 15–42. Academic Press, 1988.

[22] G. Royle. Recent results on chromatic and flow roots of graphs and matroids. In *Surveys in Combinatorics 2009, S. Huczynska, JD Mitchell, CM Roney-Dougal eds*, pages 289–327. Cambridge University Press, Cambridge, 2009.

[23] A. D. Sokal. Chromatic roots are dense in the whole complex plane. *Combin. Probab. Comput.*, **13**:221–261, 2004.

[24] C. Thomassen. The zero-free intervals for chromatic polynomials of graphs. *Combin. Probab. Comput.*, **6**:497–506, 1997.

[25] C. Thomassen. Chromatic roots and hamiltonian paths. *J. Combin. Theory Ser. B*, **80**:218–224, 2000.

[26] I. Tomescu. Chromatic coefficients of linear uniform hypergraphs. *J. Combin. Theory Ser. B*, **72**:229–235, 1998.

[27] I. Tomescu. Sunflower hypergraphs are chromatically unique. *Discrete Math.*, **285**:355–357, 2004.

[28] I. Tomescu. On the chromaticity of sunflower hypergraphs SH(n, p, h). *Discrete Math.*, **307**:781–786, 2007.

[29] I. Tomescu. Hypergraphs with pendant paths are not chromatically unique. *Discuss. Math. Graph Theory*, **34**:23–29, 2014.

[30] I. Tomescu and S. A. Bokhary. Some properties of chromatic coefficients of linear uniform hypergraphs. *Graphs Combin.*, **25**:639, 2009.

[31] W. T. Tutte. Chromials. In *Hypergraph Seminar*, pages 243–266. Springer, 1974.

[32] V. I. Voloshin. The mixed hypergraphs. *Comput. Sci. J. Moldova*, **1**:1, 1993.

[33] V. I. Voloshin. On the upper chromatic number of a hypergraph. *Australas. J. Combin.*, **11**:25–45, 1995.

[34] V. I. Voloshin. *Coloring Mixed Hypergraphs: Theory, Algorithms, and Applications*. Amer. Math. Soc., 2002.

[35] M. Walter. Some results on chromatic polynomials of hypergraphs. *Electron. J. Combin.*, **16**:R94, 2009.

[36] H. Whitney. A logical expansion in mathematics. *Bull. Amer. Math. Soc.*, **38**:572–579, 1932.

[37] D. R. Woodall. Zeros of chromatic polynomials. In *Combinatorial Surveys: Proceedings of the Sixth British Combinatorial Conference*, pages 199–223, 1977.

[38] D. R. Woodall. The largest real zero of the chromatic polynomial. *Discrete Math.*, **172**:141–153, 1997.

[39] R. Zhang and F. Dong. Properties of chromatic polynomials of hypergraphs not held for chromatic polynomials of graphs. *European J. Combin.*, **64**:138–151, 2017.