APPLYING LEARNING THEORIES TO THE TEACHING OF MATHEMATICS

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Introduction

There are many learning theories which seek to explain how learning takes place in the human mind. Although the educational psychologists are by no means agreed on how students learn and which approaches are best in which circumstances, many of the principles can however be applied to the teaching of mathematics. In fact, learning theories of Piaget, Bruner, Gagne, Dienes, Ausubel and Skemp are normally introduced to trainee teachers to provide theoretical foundations upon which they would base the planning of their lessons. These theories are briefly summarised in Cheung [1] and more thoroughly discussed in Orton [2].

The purpose of this paper is neither to enter into debates on which theory best describes undergraduate learning nor to try to synthesise the diverse theories into one composite model. It will attempt to draw from the various theories some principles which seem relevant and applicable to teaching of undergraduate mathematics.

Discussion

1. Constructivity

The cognitive psychologists, Piaget, Dienes and Bruner believe that students cannot simply be given concepts but need to construct concepts for their own through their experiences. Although the three abovementioned dealt primarily with younger children who are unable to think in abstractions, the problem of merely memorising rules without true understanding of concepts is not uncommon among undergraduates. In fact, one of the difficulties in learning mathematics, especially at tertiary levels, is the abstract nature of mathematics. Mathematicians tend to live in a world of the imagination, of symbols and abstract structures which are removed from reality. Students who cannot make sense of the concepts often resort to mechanical application of techniques and rules.

It is thus generally agreed that for meaningful learning to take place, students need to fit new concepts into existing mental structures. The theories are however not agreed on how this is to be done. While Dienes and Bruner propound exploration and discovery of the concepts, this experiential method is not so applicable in the case of older students who are able to reason in the abstract. Ausubel believes that due to the rich mental structures already present in these students, one is able to anchor new concepts on already understood concepts. He thus proposes that clear explanations building upon existing structures provide a more cost effective method of acquiring new concepts. The university way of teaching concepts is often through definitions which are precise and exclude ambiguity. However, such an approach can be very abstract in nature and the learner may not have maturity or sufficient mathematical background to have a clear understanding of the concept. The wiser teaching strategy would probably be a combination of the experiential with the analytical e.g. while one may define force as that which causes change in momentum, the experience of a push may help the student understand his mechanics in a more real way. In the case of topics where one cannot use real world experiences, the lecturer often backs up definitions with examples in order that the concepts can be better understood through examples which illustrate them. These examples are absolutely necessary in helping the students to construct the concept for themselves. The selection of examples will be discussed later in this paper.

2. Explanation

According to Ausubel's theory, clear exposition can be a most effective teaching strategy and in fact exposition is the main if not only mode in which mathematics lectures are conducted at university level. However, as the learning of mathematics goes far beyond a simple passing on of facts but requires the re-construction of concepts in the learners' minds, the lecturer does not simply tell but in fact explains so as to invite the learners to agree with the development of the concept or the theorem. As logic is the essence of mathematics, so mathematical communication is the discourse of reason. The lecturer thus has to explain clearly in order that the student can follow the logic of his argument. Some of the important principles which apply to good explanation are the following:

2.1 Proceeding from the known to the unknown.

Mathematics is a hierachical subject and concepts and principles are often based on other concepts and on relations between the concepts. Gagne proposes a careful analysis of prerequisite concepts to ensure that further teaching is based on sound foundations. In fact, this seems to be obvious and is taken into consideration by anyone planning a curriculum. For example, one must teach limits before one teaches continuity for the understanding of continuity depends on the concept of limit. Lecturers should also take care when using new notation to link these to previously used but more familiar notation. For example, in first year analysis or calculus, one tends to use $\{f(x+h) - f(x)\}/h$ in the definition of the derivative because it is more suitable at this level. However, by explaining that it is the same as the more familiar $\delta y/\delta x$, there will be less confusion due to unfamiliar notation.

2.2 Obtaining Feedback.

If it is the desire to build upon what is known, the greater problem is the lack of feedback on whether students have understood prerequisite concepts. The lecture-tutorial system especially when the tutorial sessions are spent on providing solutions rather than ascertaining whether students have understood the material covered is certainly not conducive to obtaining feedback. Lecturers are often shocked and horrified to find out from the end-of-the-year examinations that concepts are poorly understood and theorems are not understood let alone correctly applied. Continual feedback from students is necessary and the means of obtaining feedback is through tutorial sessions where tutors can question students on lecture material. Asking challenging questions force the students to think about the lecture material and experienced tutors who know where students usually have misconceptions can question students especially on known weak areas. Linked to the above point on using examples to illustrate mathematical concepts, students could be asked to construct their own examples. This also promotes creative thinking in addition to ensuring that they have truly understood the concept.

2.3 Using illustrations and linkages.

It was discussed above how examples could be used to provide "concretisation" of abstractions. It must be added here that for examples to better illustrate the definition, the lecturer needs to draw the learners' attention to <u>how</u> exactly the example fits each condition of the explanation. For example, in using the real numbers to illustrate a field, the lecturer should show how each field axiom is satisfied by the field of real numbers. For another example, it is not sufficient to say that marriage is a one-to-one correspondence between the set of all husbands and the set of all wives but one should explain how and qualify that the example satisfies the one-to-one property only in the countries where bigamy is illegal. The two previous examples are very simple ones but this idea of linking the example to each condition in the definition is even more crucial when the concept involves conditions and other concepts which are abstract and difficult in themselves.

Explanations can also become clearer when linkages are made between different modes of representation such as the diagrammatic or graphical as opposed to the analytic or algebraic. Bruner's theory states that there are three modes of representation: the enactive, the iconic and the symbolic. By offering explanations in more than one mode where applicable, especially in cases where a result can be interpreted pictorially, one would be able to make the situation clearer and more comprehensive as well as show students the relationship between the various interpretations of the result. While this seems to be an obvious point and many mathematics lecturers do illustrate their points with relevant diagrams, one should not grudge spending time on clear diagrams nor disdain as childish or low level the use teaching aids, transparency overlays etc if such visuals can help our students' understand better.

2.4 Using overviews

One lament students make about proofs is that they can follow the proof step by step but do not see how the whole proof is constructed. Part of this is because the numerous details and logical steps obscure the overall structure of the proof. This is especially the case when the proofs are very long and depend on lemmas which have to be proved first (for example, the proof of the Central Limit Theorem). In the presentation of proofs, lecturers tend to begin at the beginning and proceed logically to the end. This leaves the students without a "helicopter view" of the proof. In fact, this can also be said for a topic wherein one knows the various concepts and the various theorems but have no overview of the landscape and how all these stand in relation to each other. Organisation of exposition into main points and the giving of an overview before and a summary after a proof or a topic are helpful to the students in this respect.

3. Examples

According to Skemp [3], concepts are built upon other concepts or abstracted from reality. Although higher level mathematics depends on definitions to ensure that the concept is correctly put across, it is nevertheless the examples that lend a sense of "reality" to the concept defined. As Dienes strongly advocated that concepts be abstracted from experience, he propounded that such experiences should have the mathematical variability so that what is abstracted is the essence of the concept and that no unnecessary attribute is allowed to inadvertently creep into the understanding of the concept.

An example of how unnecessary attributes can be subsumed was apparent when a class of graduate trainee teachers were asked to define a function f on the reals given the values of f(1), f(2), f(3) and f(0). Six of the class of 31 tried to find polynomial functions despite the fact that they have seen examples of discontinuous and non-differentiable functions at university level. This is because at 'A' levels and below, although the notion of a function as a mapping is taught, almost all functions encountered are polynomials, trigonometric or exponential and as they had less though more recent experience with discontinuous functions, they unconsciously restricted themselves to those examples of functions they encountered more often. (See Vinner & Dreyfus [4].)

Thus it is the examples rather than the definitions which tend to form the concept image carried in the learner's mind and Dienes' variability principle warns us to be careful in having as wide a range as possible of examples to illustrate and to "concretise" the definitions we seek to teach in our lectures. In fact, non-examples which could be confused as examples need to be especially highlighted.

4. Problem-solving

Mathematics educators are concerned that mathematics students are pre-occupied with memorising facts, practising techniques and regurgitating these in the examinations. Except for the best ones, students on the other hand lament that although they are able to follow the proofs in the lectures or understand how to do the problems when the tutor explains, they are unable to write proofs themselves and are often at a lost when new problems are encountered. At times, they do not even understand the problem, let alone begin to solve it.

The solving of problems and in fact the posing of problems is the absorbing activity all mathematicians do. Polya [5] proposed a model where the main steps in solving problems are (1) Understand the problem (2) Devise a plan

(3) Carry out the plan and (4) Review. Problem-solving strategies will be introduced to schools as from 1992 in the new Primary Mathematics syllabus and the lower secondary syllabus. However, as the 'O' levels and the 'A' levels remain unchanged, it is feared that such strategies will not be applied by the students at university level unless students are encouraged to do so. Tutorial sessions should not degenerate into solution presenting sessions by the tutors but should be discussion sessions where students are led to restate the problems, examine conditions given, search for concepts to be applied, review their solutions for logical consistency, etc. Students could also be asked to pose their own problems or extend given ones by modifying conditions or by asking "what if" questions. Conscious application of such strategies can enable our students to solve more mathematics problems on their own and to think mathematically in new situations. Naturally, such approaches are time-consuming but here is the dilemma of quality versus quantity which every department has to resolve for itself.

Conclusion

The above has discussed how some principles from the theories of mathematics learning can be applied to university level teaching. One aspect which has not been discussed is that of motivation. Whereas primary and secondary school teachers work very hard to make their lessons interesting in order to motivate the unmotivated students, university students are a self-motivated lot even if their motivation may not be intrinsic i.e. the "love" for the subject. This does not, however, imply that lecturers are free from the the task of stimulating students' interest. On the contrary, academics who by virtue of their chosen career must have a genuine interest in the subject should be the best people to make their enthusiasm contagious so that students will be inspired to seek even more mathematical truths for themselves.

There are some for whom teaching is a gift they are born with and they teach well effortlessly to the good fortune of their students. For others, it takes firstly, conscious awareness of what constitutes good teaching and secondly, careful effort in preparation and delivery to improve their natural teaching ability. It may well be a creed for all who teach to constantly remember that unless their students have learnt, they have not truly taught.

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