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Bell inequalities for three particles

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We present tight Bell inequalities expressed by probabilities for three four- and five-dimensional systems. A tight structure of Bell inequalities for three d-dimensional systems (qudits) is proposed. Some interesting Bell inequalities of three qubits reduced from those of three qudits are also studied.

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I. INTRODUCTION

That no local and realistic theory agrees with all predictions of quantum mechanics was shown by Bell in 1964 [1] through the violations of certain constraints. Local realism imposes constraints in the form of Bell inequalities on statistical correlations of measurements on multiparticles. Quantum mechanics predicts violations of such Bell inequalities. The original Bell inequality and the subsequent famous Clauser-Horne-Shimony-Holt (CHSH) inequality [2], the latter being cast into a form more amenable for experimental verification, were formulated for the simplest composite quantum system: namely, a system of two qubits.

Bell inequalities which eliminate a local realistic description are of importance not only for fundamental research, but for identifying ultimate resources for quantum-information processing. It was shown that there is a direct link between the security of quantum cryptography and the violation of Bell inequalities [3,4]. Collins *et al.* [5] found a tight Bell inequality for two arbitrary d-dimensional systems (or two qudits) in terms of joint probabilities; hereafter, we call it the Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequality. For $d=2$, the CGLMP inequality reduces to the CHSH inequality. Alternatively, for N particles of two dimensions (called an N qubit), it was shown that there exists a general Bell inequality which is a sufficient and necessary condition for N-body correlations to be describable in a local and realistic theory based on two local settings for each observer [6]. However, Bell inequalities for N ($N>2$) entangled d-dimensional ($d>2$) quantum systems are not so well formulated as those for two-qudits or N-qubits. Only recently have the problems been solved partly in the case of three three-dimensional particles in [7]. The authors developed a coincidence Bell inequality in terms of probabilities. For general N ($N>2$) entangled d-dimensional quantum systems, no Bell inequality for either probabilities or correlation functions has been presented until now, although GHZ paradox has been generalized to systems of N qudits [8].

Even for N-qubit systems, there remains one problem with the Bell inequality; that is, do all pure entangled states violate Bell inequalities for correlation functions? In other words, it is whether the theorem of Gisin [9,10] can be generalized to N qubits or not. It is found that there is a family of pure entangled states of N qubits which do not violate all Bell inequalities [11]. For three qubits, we have proposed a Bell inequality to solve the problem [12].

In this work, we present the tight Bell inequalities expressed by probabilities for three four- and five-dimensional systems. A tight structure of Bell inequalities for three d-dimensional systems (qudits) is proposed. Some interesting Bell inequalities for three qubits reduced from those of three qudits are also studied.

II. BELL INEQUALITIES FOR THREE *d*-LEVEL SYSTEMS

Local realism cannot exhibit arbitrary correlations. The constraints that local realistic correlations must obey can be written in the form of Bell inequalities. For three d-dimensional systems with an arbitrary value of d, some efforts have been given to develop Bell inequalities recently. The first step came in 1990 with a paper of Mermin [13] in which he derived a Bell inequality for arbitrary N-qubit states; quantum mechanics violates this inequality by an amount that grows with N. This result clearly gives us a first three-qubit Bell inequality in a correlation form,

$$Q_{112}+Q_{121}+Q_{211}-Q_{222}\leqslant 2. \tag{1}$$

It can be expressed in terms of probabilities

$$\begin{aligned}P(a_1+b_1+c_2=0)&-P(a_1+b_1+c_2=1)+P(a_1+b_2+c_1=0)\\&-P(a_1+b_2+c_1=1)+P(a_2+b_1+c_1=0)\\&-P(a_2+b_1+c_1=1)-P(a_2+b_2+c_2=0)\\&+P(a_2+b_2+c_2=1)\leqslant 2.\end{aligned} \tag{2}$$

This Bell inequality is maximally violated by the three-qubit Greenberger-Horne-Zeilinger (GHZ) state $|\psi_2\rangle=\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle)$. But for the generalized GHZ states $|\psi_2\rangle_{GHZ}=\cos\xi|000\rangle+\sin\xi|111\rangle$, there exists one region $\xi\in(0,\pi/12]$ in which the Bell inequality is not violated.

The second step is due to Ref. [7]. The authors developed a three-qutrit Bell inequality involving probabilities which can be given in an alternative form

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$$\begin{aligned}
&-P(a_1+b_1+c_1=0)-P(a_1+b_1+c_1=1)\\
&\quad+2P(a_1+b_1+c_1=2)+P(a_1+b_1+c_2=0)\\
&\quad-2P(a_1+b_1+c_2=1)+P(a_1+b_1+c_2=2)\\
&\quad+P(a_1+b_2+c_1=0)-2P(a_1+b_2+c_1=1)\\
&\quad+P(a_1+b_2+c_1=2)+P(a_2+b_1+c_1=0)\\
&\quad-2P(a_2+b_1+c_1=1)+P(a_2+b_1+c_1=2)\\
&\quad+2P(a_1+b_2+c_2=0)-P(a_1+b_2+c_2=1)\\
&\quad-P(a_1+b_2+c_2=2)+2P(a_2+b_1+c_2=0)\\
&\quad-P(a_2+b_1+c_2=1)-P(a_2+b_1+c_2=2)\\
&\quad+2P(a_2+b_2+c_1=0)-P(a_2+b_2+c_1=1)\\
&\quad-P(a_2+b_2+c_1=2)-2P(a_2+b_2+c_2=0)\\
&\quad-2P(a_2+b_2+c_2=1)+4P(a_2+b_2+c_2=2)\leqslant 6.
\end{aligned}\tag{3}$$

The above inequality is maximally violated by the three-qutrit GHZ state $|\psi_3\rangle=\frac{1}{\sqrt{3}}(|000\rangle+|111\rangle+|222\rangle)$. It is worth mentioning that both inequalities (2) and (3) are tight.

Here we give the third step. Our approach of constructing a new Bell inequality for tripartite four-dimensional systems is based on the Gedanken experiment. There are three separated observers, denoted by A, B, and C hereafter; each can carry out two possible local measurements A_1 or A_2 for A, B_1 or B_2 for B, and C_1 or C_2 for C, respectively. Each measurement may have four possible outcomes, labeled by 0, 1, 2, and 3. We denote the observable X_i measured by party X and the outcome x_i with $X=A,B,C$ $(x=a,b,c)$. A local realistic theory can be described by 8×64 probabilities. Here we denote the joint probability $P(a_i+b_j+c_k=r)$ that the measurements A_i, B_j, and C_k have outcomes that differ, modulo 4, by r:

$$P(a_i+b_j+c_k=r)=\sum_{a,b=0,1,2,3}P(a_i=a,b_j=b,c_k=r-a-b).\tag{4}$$

Some of the local realistic constraints are trivial, such as normalization and the no-signaling conditions which are not violated by quantum predictions. Only the nontrivial inequality, which is not true for quantum mechanics, is of use for checking whether we can describe quantum correlations by a classical model. The new Bell inequality for three four-dimensional systems reads

$$\begin{aligned}
&-5P(a_1+b_1+c_1=0)+P(a_1+b_1+c_1=1)\\
&\quad+3P(a_1+b_1+c_1=2)+P(a_1+b_1+c_1=3)\\
&\quad+3P(a_1+b_1+c_2=0)-7P(a_1+b_1+c_2=1)\\
&\quad+3P(a_1+b_1+c_2=2)+P(a_1+b_1+c_2=3)\\
&\quad+3P(a_1+b_2+c_1=0)-7P(a_1+b_2+c_1=1)\\
&\quad+3P(a_1+b_2+c_1=2)+P(a_1+b_2+c_1=3)\\
&\quad+3P(a_2+b_1+c_1=0)-7P(a_2+b_1+c_1=1)\\
&\quad+3P(a_2+b_1+c_1=2)+P(a_2+b_1+c_1=3)\\
&\quad+3P(a_1+b_2+c_2=0)+P(a_1+b_2+c_2=1)\\
&\quad-5P(a_1+b_2+c_2=2)+P(a_1+b_2+c_2=3)\\
&\quad+3P(a_2+b_1+c_2=0)+P(a_2+b_1+c_2=1)\\
&\quad-5P(a_2+b_1+c_2=2)+P(a_2+b_1+c_2=3)\\
&\quad+3P(a_2+b_2+c_1=0)+P(a_2+b_2+c_1=1)\\
&\quad-5P(a_2+b_2+c_1=2)+P(a_2+b_2+c_1=3)\\
&\quad-P(a_2+b_2+c_2=0)-3P(a_2+b_2+c_2=1)\\
&\quad-P(a_2+b_2+c_2=2)+5P(a_2+b_2+c_2=3)\leqslant 12.
\end{aligned}\tag{5}$$

That the maximum value of the left-hand side of inequality (5) for local theories is 12 can be given in the following sense. By using $\Sigma_{r=0}^{3}P(a_i+b_j+c_k=r)=1$, inequality (5) is reformed as

$$\begin{aligned}
&-3P(a_1+b_1+c_1=0)+P(a_1+b_1+c_1=2)\\
&\quad-5P(a_1+b_1+c_2=1)-P(a_1+b_1+c_2=3)\\
&\quad-5P(a_1+b_2+c_1=1)-P(a_1+b_2+c_1=3)\\
&\quad-5P(a_2+b_1+c_1=1)-P(a_2+b_1+c_1=3)\\
&\quad+P(a_1+b_2+c_2=0)-3P(a_1+b_2+c_2=2)\\
&\quad+P(a_2+b_1+c_2=0)-3P(a_2+b_1+c_2=2)\\
&\quad+P(a_2+b_2+c_1=0)-3P(a_2+b_2+c_1=2)\\
&\quad-P(a_2+b_2+c_2=1)+3P(a_2+b_2+c_2=3)\leqslant 0.
\end{aligned}\tag{6}$$

To beat the bound 0, the terms $P(a_1+b_1+c_1=2)$ and $P(a_2+b_2+c_2=3)$ are taken equal to 1 first. This means that $a_1+b_1+c_1+a_2+b_2+c_2=5$. Among the remaining terms, we take $P(a_1+b_1+c_2=3)$, $P(a_1+b_2+c_1=3)$, and $P(a_2+b_1+c_1=3)$ equal to 1 to maximize the value of the left-hand side of inequality (6). As a result, $a_2+b_2+c_1=2$, $a_2+b_1+c_2=2$ and $a_1+b_2+c_2=2$ according to the constraint $a_1+b_1+c_1+a_2+b_2+c_2=5$. So $P(a_1+b_1+c_2=1)=P(a_1+b_2+c_1=1)=P(a_2+b_1+c_1=1)=0$, $P(a_2+b_2+c_1=2)=P(a_2+b_1+c_2=2)=P(a_1+b_2+c_2=2)=1$, and $P(a_2+b_2+c_1=0)=P(a_2+b_1+c_2=0)=P(a_1+b_2+c_2=0)=0$. Therefore we have $0+1-0-1-0-1-0-1+0-3+0-3+0-3-0+3=-8\leqslant 0$. If initially the terms $P(a_1+b_1+c_1=0)$ and $P(a_2+b_2+c_2=3)$ are taken equal to 1 first. This means that $a_1+b_1+c_1+a_2+b_2+c_2=3$. Among the remaining terms, we take $P(a_1+b_1+c_2=3)$, $P(a_1+b_2+c_1=3)$, and $P(a_2+b_1+c_1=3)$ equal to 1 to maximize the value of the left-hand side of inequality (6). As a result, $a_2+b_2+c_1=0$, $a_2+b_1+c_2=0$, and $a_1+b_2+c_2=0$ according to the constraint $a_1+b_1+c_1+a_2+b_2+c_2=3$. So $P(a_1+b_1+c_2=1)=P(a_1+b_2+c_1=1)=P(a_2+b_1+c_1=1)=0$, $P(a_2+b_2+c_1=2)=P(a_2+b_1+c_2=2)=P(a_1+b_2+c_2=2)=0$, and $P(a_2+b_2+c_1=0)=P(a_2+b_1+c_2=0)=P(a_1+b_2+c_2=0)=1$. Therefore we have $-3+0-0-1-0-1-0-1+1-0+1-0+1-0-0+3=0\leqslant 0$. Therefore, after some simple and patient calculations, it can be shown that inequality (6) is always bounded by 0 in a local realistic model. Furthermore, the Bell inequality (5) is a tight inequality for three four-dimensional systems [14].

TABLE I. The values of the quantum joint probabilities in inequality (5) with appropriate angle settings.

$p(a_1+b_1+c_1=0)$	$p(a_1+b_1+c_1=1)$	$p(a_1+b_1+c_1=2)$	$p(a_1+b_1+c_1=3)$
0	1/6	2/3	1/6
$p(a_1+b_1+c_2=0)$	$p(a_1+b_1+c_2=1)$	$p(a_1+b_1+c_2=2)$	$p(a_1+b_1+c_2=3)$
1/2	0	1/2	0
$p(a_1+b_2+c_1=0)$	$p(a_1+b_2+c_1=1)$	$p(a_1+b_2+c_1=2)$	$p(a_1+b_2+c_1=3)$
1/2	0	1/2	0
$p(a_2+b_1+c_1=0)$	$p(a_2+b_1+c_1=1)$	$p(a_2+b_1+c_1=2)$	$p(a_2+b_1+c_1=3)$
1/2	0	1/2	0
$p(a_1+b_2+c_2=0)$	$p(a_1+b_2+c_2=1)$	$p(a_1+b_2+c_2=2)$	$p(a_1+b_2+c_2=3)$
2/3	1/6	0	1/6
$p(a_2+b_1+c_2=0)$	$p(a_2+b_1+c_2=1)$	$p(a_2+b_1+c_2=2)$	$p(a_2+b_1+c_2=3)$
2/3	1/6	0	1/6
$p(a_2+b_2+c_1=0)$	$p(a_2+b_2+c_1=1)$	$p(a_2+b_2+c_1=2)$	$p(a_2+b_2+c_1=3)$
2/3	1/6	0	1/6
$p(a_2+b_2+c_2=0)$	$p(a_2+b_2+c_2=1)$	$p(a_2+b_2+c_2=2)$	$p(a_2+b_2+c_2=3)$
1/18	0	1/18	8/9

Let us now consider the maximum value that can be attained for inequality (5) for quantum measurements on an entangled quantum state. First, we specify the quantum state and measurement. The initial state is a natural generalization of bipartite maximally entangled state to three four-level systems,

$$|\psi_4\rangle = \frac{1}{2}(|000\rangle + |111\rangle + |222\rangle + |333\rangle). \tag{7}$$

Consider a Gedanken experiment in which A, B, and C measure observables defined by unbiased symmetric multiport beam splitters [15] on $|\psi\rangle$. The unbiased symmetric multiport beam splitter is an optical device with d input and d output ports. In front of every input port there is a phase shifter that changes the phase of the photon entering the given port. If a phase shifter in some input port is set to zero and a photon enters the device through this port, then it has an equal chance of leaving the device through any output port. The phase shifters can be changed by the observers; they represent the local macroscopic parameters available to the observers. The matrix elements of an unbiased symmetric multiport beam splitter are given by $U_{kl}(\vec{\phi}) = \frac{1}{\sqrt{d}}\alpha^{kl}\exp(i\phi^l)$, where $\alpha = \exp(\frac{2i\pi}{d})$ and $\phi^l(l=0,1,2,\ldots,d-1)$ are the settings of the appropriate phase shifters; for convenience, we denote them as a d-dimensional vector $\vec{\phi} = (\phi^0,\phi^1,\phi^2,\ldots,\phi^{d-1})$. For four-dimensional systems, $d=4$.

The quantum prediction for the probabilities of obtaining the outcome (a,b,c) is then given as

$$P(a_i=a,b_j=b,c_k=c) = |\langle abc|U(\vec{\phi}_A)\otimes U(\vec{\phi}_B) \otimes U(\vec{\phi}_C)|\psi_4\rangle|^2. \tag{8}$$

Thus the quantum analogue of the joint probability can be easily calculated:

$$\begin{aligned} P(a_i+b_j+c_k=r) = \frac{1}{16}\Bigg[& 4 + 2\cos\left(\varphi^1-\varphi^0+\frac{\pi}{2}r\right) \\ & + 2\cos(\varphi^2-\varphi^0+\pi r) \\ & + 2\cos\left(\varphi^2-\varphi^1+\frac{\pi}{2}r\right) \\ & + 2\cos\left(\varphi^3-\varphi^0+\frac{3\pi}{2}r\right) \\ & + 2\cos(\varphi^3-\varphi^1+\pi r) \\ & + 2\cos\left(\varphi^3-\varphi^2+\frac{\pi}{2}r\right)\Bigg], \end{aligned} \tag{9}$$

where $\varphi^i = \phi^i_A + \phi^i_B + \phi^i_C$ $(i=0,1,2,3)$. In order to look for the maximal violation of the inequality, we choose the optimal settings as follows: $\vec{\phi}_{A1} = \vec{\phi}_{B1} = \vec{\phi}_{C1} = (0,\frac{1}{3}\arccos(-\frac{1}{3}),\frac{1}{3}\arccos(-\frac{1}{3})-\frac{\pi}{3},\frac{\pi}{3})$ and $\vec{\phi}_{A2} = \vec{\phi}_{B2} = \vec{\phi}_{C2} = (0,\frac{1}{3}\arcsin\frac{7}{9},\frac{1}{3}\arcsin\frac{7}{9}+\frac{\pi}{6},-\frac{\pi}{6})$. Numerical results show that for this choice, all the probability terms have definite values as listed in Table I. Putting them into the left-hand side of inequality (5), we arrive at $2\frac{1}{6}+3\frac{2}{3}+3(3\frac{1}{2}+3\frac{1}{2})+3(3\frac{2}{3}+2\frac{1}{6})-2\frac{1}{18}+5\frac{8}{9}=\frac{68}{3}>12$.

In Ref. [16], a proposal was made to measure the strength of violation of local realism by the minimal amount of noise that must be added to the system in order to hide the nonclassical character of the observed correlations. This is equivalent to a replacement of the pure state $|\psi\rangle\langle\psi|$ by the mixed state $\rho(F)$ of the form $\rho(F)=(1-F)|\psi\rangle\langle\psi|+\frac{F}{64}I\otimes I\otimes I$, where I is an identity matrix and $F(0\leqslant F\leqslant 1)$ is the amount of noise present in the system. For $F=0$, a local realistic description does not exist, whereas it does for $F=1$. Therefore, there exists some threshold value of F, denoted by F_{thr}, such that for every $F\leqslant F_{thr}$, a local and realistic description does not exist. The threshold fidelity for the three four-level systems is determined by $(1-F_{thr})\frac{68}{3}=12$: namely, $F_{thr}=\frac{8}{17}=0.4706$.

Similarly, we propose a Bell inequality for three five-dimensional systems based on the Gedanken experiment:

$$\begin{aligned}
&-2P(a_1+b_1+c_1=0)+P(a_1+b_1+c_1=1)\\
&\quad+P(a_1+b_1+c_1=4)+P(a_1+b_1+c_2=0)\\
&\quad-2P(a_1+b_1+c_2=2)+P(a_1+b_1+c_2=4)\\
&\quad+P(a_1+b_2+c_1=0)-2P(a_1+b_2+c_1=2)\\
&\quad+P(a_1+b_2+c_1=4)+P(a_2+b_1+c_1=0)\\
&\quad-2P(a_2+b_1+c_1=2)+P(a_2+b_1+c_1=4)\\
&\quad+P(a_1+b_2+c_2=0)+P(a_1+b_2+c_2=3)\\
&\quad-2P(a_1+b_2+c_2=4)+P(a_2+b_1+c_2=0)\\
&\quad+P(a_2+b_1+c_2=3)-2P(a_2+b_1+c_2=4)\\
&\quad+P(a_2+b_2+c_1=0)+P(a_2+b_2+c_1=3)\\
&\quad-2P(a_2+b_2+c_1=4)-2P(a_2+b_2+c_2=1)\\
&\quad+P(a_2+b_2+c_2=3)+P(a_2+b_2+c_2=4)\leqslant 4,
\end{aligned} \tag{10}$$

which is satisfied by the local and realistic theories. Also, the Bell inequality (10) is also tight [14].

Using a specified quantum state and measurement, we calculate the maximum value that can be attained for the inequality (10). The considered state is a natural generalization of bipartite maximally entangled state to three five-level systems,

$$|\psi_5\rangle=\frac{1}{\sqrt{5}}(|000\rangle+|111\rangle+|222\rangle+|333\rangle+|444\rangle). \tag{11}$$

The measurement is also based on an unbiased symmetric multiport beam splitter with $d=5$, and the quantum prediction for the probabilities of obtaining the outcome (a,b,c) is then given as

$$\begin{aligned}
P(a_i=a,b_j=b,c_k=c)=&|\langle abc|U(\vec{\phi}_A)\otimes U(\vec{\phi}_B)\\
&\otimes U(\vec{\phi}_C)|\psi_5\rangle|^2.
\end{aligned} \tag{12}$$

Thus the quantum analog of the joint probability is given as

$$\begin{aligned}
P(a_i+b_j+c_k=r)=\frac{1}{25}\Bigg[&5+2\cos\left(\varphi^1-\varphi^0+\frac{2\pi}{5}r\right)+2\cos\left(\varphi^2-\varphi^0+\frac{4\pi}{5}r\right)+2\cos\left(\varphi^2-\varphi^1+\frac{2\pi}{5}r\right)+2\cos\left(\varphi^3-\varphi^0+\frac{6\pi}{5}r\right)\\
&+2\cos\left(\varphi^3-\varphi^1+\frac{4\pi}{5}r\right)+2\cos\left(\varphi^3-\varphi^2+\frac{2\pi}{5}r\right)+2\cos\left(\varphi^4-\varphi^0+\frac{8\pi}{5}r\right)+2\cos\left(\varphi^4-\varphi^1+\frac{6\pi}{5}r\right)\\
&+2\cos\left(\varphi^4-\varphi^2+\frac{4\pi}{5}r\right)+2\cos\left(\varphi^4-\varphi^3+\frac{2\pi}{5}r\right)\Bigg],
\end{aligned} \tag{13}$$

where $\varphi^i=\phi_A^i+\phi_B^i+\phi_C^i$ $(i=0,1,2,3,4)$. Numerical results show that for the angle settings $\vec{\phi}_{A1}=\vec{\phi}_{B1}=\vec{\phi}_{C1}=(0,\beta_1,\beta_2,-\beta_2,-\beta_1)$ and $\vec{\phi}_{A2}=\vec{\phi}_{B2}=\vec{\phi}_{C2}=(0,\beta_1+\frac{\pi}{5},\beta_2+\frac{2\pi}{5},-\beta_2-\frac{2\pi}{5},-\beta_1-\frac{\pi}{5})$, where $\cos(3\beta_1)-\cos(3\beta_2)=\frac{1}{2}$, the maximum value of the left-hand side of inequality (10) attained is 6.722 16. The threshold fidelity for the three five-level systems is determined by $(1-F_{thr})\times 6.72216=4$: namely, $F_{thr}=0.404\,95$.

Given the above known Bell inequalities for three particles with $d=2,3,4,5$, it is worth noting that such tight inequalities exhibit perfect symmetries. One symmetry is that these Bell inequalities are symmetric under the permutations of subsystems A, B, and C. The second symmetry is that they can be expressed in a general form based on which the structure of tight Bell inequalities is suggested as

$$\frac{1}{d(d-1)}\sum_{ijk}\sum_{r=0}^{d-1}f_{d;ijk}^{r}P(a_i+b_j+c_k=r)\leqslant 1, \tag{14}$$

with $-1\leqslant\frac{f_{d;ijk}^{r}}{d(d-1)}\leqslant 1$. It is worth noting that the tight Bell inequalities for two qudits (i.e., the CGLMP inequality; see [5]) can be written in a similar form

$$\frac{1}{d-1}\sum_{ij}\sum_{r=0}^{d-1}f_{d;ij}^{r}P(a_i+b_j=r)\leqslant 1, \tag{15}$$

with $-1\leqslant\frac{f_{d;ij}^{r}}{d-1}\leqslant 1$. The coefficients $f_{d;ijk}^{r}$ and $f_{d;ij}^{r}$ are integers or half integers. Note that these inequalities are fulfilled with displacement of probabilities. In other words, for a known number m, where $m\leqslant d-1$, the above inequalities are still true for a local realistic description by replacing $P(a_i+b_j+c_k=r)$ with $P(a_i+b_j+c_k=r+m)$, which is the third symmetry of the set of Bell inequalities for three particles.

III. INTERESTING BELL INEQUALITIES OF THREE QUBITS REDUCED FROM THOSE OF THREE QUDITS

In 1991 Gisin presented a theorem, which states that *any* pure entangled state of two particles violates a Bell inequality for two-particle correlation functions [9,10]. Recent investigations show a surprising result, that there exists a family of pure entangled N-qubit states that does not violate any Bell inequality for N-particle correlations for the case of a standard Bell experiment on N qubits [17]. This family is the family of generalized GHZ states given by

$$|\psi_2\rangle_{GHZ} = \cos\xi|0\cdots0\rangle + \sin\xi|1\cdots1\rangle, \tag{16}$$

with $0\leqslant\xi\leqslant\pi/4$. The usual GHZ states [18] are for $\xi=\pi/4$. For a three-qubit system, whose corresponding generalized GHZ state reads $|\psi_2\rangle_{GHZ}=\cos\xi|000\rangle+\sin\xi|111\rangle$, it has been shown that for the region $\xi\in(0,\pi/12]$, the inequalities given in [6] are not violated based on the standard Bell experiment. Recently, we developed a three-qubit Bell inequality which is a solution to such a problem. That is all pure entangled states of three qubits violate the Bell inequality given in [12]. Indeed Bell inequalities are sensitive to the presence of noise and above a certain amount of noise the Bell inequalities will cease to be violated by a quantum system. However, it seems that the inequality in Ref. [12] is not good enough to the resistance of noise. For the three-qubit GHZ state, the threshold visibility is $V_{GHZ}=4\sqrt{3}/9=0.7698$, and for the W state, the threshold visibility is $V_W=0.7312$. The inequality in Ref. [12] can be derived from three-qutrit Bell inequality (3). Actually any Bell inequality for tripartite d-level ($d>2$) systems may reduce to a Bell inequality for three qubits when one considers only two outcomes of measurement.

Here we present a new Bell inequality for three-qubit systems which is reduced from inequality (5):

$$\begin{aligned}
&3P(a_1+b_1+c_1=0)+P(a_1+b_1+c_1=1)\\
&\quad-5P(a_1+b_1+c_1=2)+P(a_1+b_1+c_1=3)\\
&\quad+3P(a_1+b_1+c_2=0)+P(a_1+b_1+c_2=1)\\
&\quad+3P(a_1+b_1+c_2=2)-7P(a_1+b_1+c_2=3)\\
&\quad+3P(a_1+b_2+c_1=0)+P(a_1+b_2+c_1=1)\\
&\quad+3P(a_1+b_2+c_1=2)-7P(a_1+b_2+c_1=3)\\
&\quad+3P(a_2+b_1+c_1=0)+P(a_2+b_1+c_1=1)\\
&\quad+3P(a_2+b_1+c_1=2)-7P(a_2+b_1+c_1=3)\\
&\quad-5P(a_1+b_2+c_2=0)+P(a_1+b_2+c_2=1)\\
&\quad+3P(a_1+b_2+c_2=2)+P(a_1+b_2+c_2=3)\\
&\quad-5P(a_2+b_1+c_2=0)+P(a_2+b_1+c_2=1)\\
&\quad+3P(a_2+b_1+c_2=2)+P(a_2+b_1+c_2=3)\\
&\quad-5P(a_2+b_2+c_1=0)+P(a_2+b_2+c_1=1)\\
&\quad+3P(a_2+b_2+c_1=2)+P(a_2+b_2+c_1=3)\\
&\quad-P(a_2+b_2+c_2=0)+5P(a_2+b_2+c_2=1)\\
&\quad-P(a_2+b_2+c_2=2)-3P(a_2+b_2+c_2=3)\leqslant 12.
\end{aligned}\tag{17}$$

The inequality can be expressed in terms of correlation functions

$$\begin{aligned}
&-E(A_1B_1C_1)+E(A_1B_1C_2)+E(A_1B_2C_1)+E(A_2B_1C_1)\\
&\quad-E(A_2B_2C_2)-E(A_1B_2)-E(A_2B_1)-E(A_2B_2)\\
&\quad-E(A_1C_2)-E(A_2C_1)-E(A_2C_2)-E(B_1C_2)-E(B_2C_1)\\
&\quad-E(B_2C_2)+E(A_1)+E(B_1)+E(C_1)\leqslant 3.
\end{aligned}\tag{18}$$

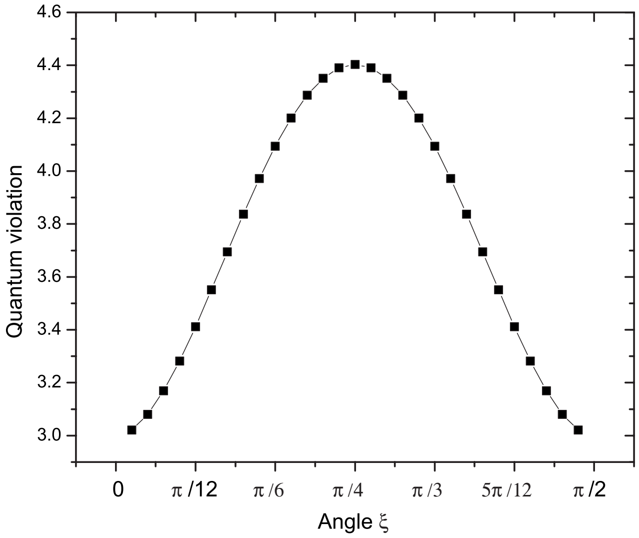

FIG. 1. Numerical results for the generalized GHZ states $|\psi_2\rangle_{GHZ}=\cos\xi|000\rangle+\sin\xi|111\rangle$, which violate inequality (18) except $\xi=0$, $\pi/2$.

The above inequality (18) includes the terms of single correlation functions; it is symmetric under the permutations of A_j, B_j, and C_j. Quantum mechanically, the above inequality is violated by all pure entangled states of three qubits. Pure states of three qubits constitute a five-parameter family, with equivalence up to local unitary transformations. This family has the following representation [19]:

$$\begin{aligned}
|\psi\rangle &= \sqrt{\mu_0}|000\rangle+\sqrt{\mu_1}e^{i\phi}|100\rangle+\sqrt{\mu_2}|101\rangle+\sqrt{\mu_3}|110\rangle\\
&\quad+\sqrt{\mu_4}|111\rangle,
\end{aligned}\tag{19}$$

with $\mu_i\geqslant 0$, $\Sigma_i\mu_i=1$, and $0\leqslant\phi\leqslant\pi$. Numerical results show that this Bell inequality for probabilities is violated by all pure entangled states of three-qubit systems. However, no analytical proof of the conclusion has been given. In the following, some special cases will be given to show that inequality (18) is violated by all pure entangled states. The first quantum state considered is a generalized GHZ state $|\psi_2\rangle_{\mathrm{GHZ}}=\cos\xi|000\rangle+\sin\xi|111\rangle$. Inequality (18) is violated by the generalized GHZ states for the whole region except $\xi=0$, $\pi/2$. For the GHZ state with $\xi=\pi/4$, the quantum violation reaches its maximum value 4.403 67. The variation of the violation with ξ is shown in Fig. 1. Another state considered is the generalized W state $|\psi\rangle_W=\sin\beta\cos\xi|100\rangle+\sin\beta\sin\xi|010\rangle+\cos\beta|001\rangle$. By fixing the value of β, the quantum violation of inequality (18) varies with ξ (see Fig. 2). Inequality (18) is violated by the generalized W states except for the cases with $\beta=\frac{\pi}{2}$, $\xi=0$, and $\xi=\frac{\pi}{2}$. The states in these cases are direct-product states which do not violated any Bell inequality. For the standard W state, quantum violation of inequality (18) approaches 4.540 86.

Hence inequality (18) is also one candidate to generalize the theorem of Gisin to three-qubit systems. One of the features of the new inequality for three qubits is that it is highly resistant to noise. Inequality (18) is violated by the general-

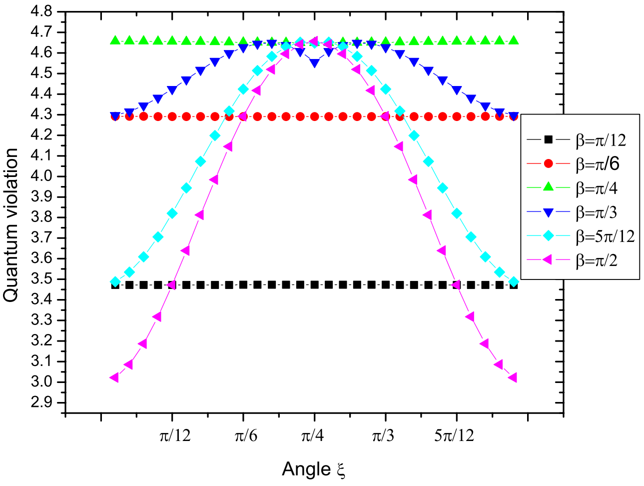

FIG. 2. (Color online) Numerical results for the generalized W states $|\psi\rangle_W = \sin\beta\cos\xi|100\rangle + \sin\beta\sin\xi|010\rangle + \cos\beta|001\rangle$ which violate inequality (18) for different ξ and β. Here the cases $\beta = \pi/12$, $\pi/6$, $\pi/4$, $\pi/3$, $5\pi/12$, and $\pi/2$ are considered.

ized GHZ state $|\psi_2\rangle_{GHZ} = \cos\xi|000\rangle + \sin\xi|111\rangle$ for the whole region; the threshold visibility is $V_{thr}^{GHZ} = 0.68125$. Inequality (18) is also violated by the W state; the threshold visibility is $V_{thr}^{W} = 0.660\,668$. We plot the variation of quantum violation for the generalized GHZ states with angle ξ for inequality (18) and the inequality given in Ref. [12]; see Fig. 3. In plotting the figure, we reform the expressions of these two inequalities as

$$\begin{aligned}\frac{1}{4}[&Q(A_1B_1C_1) - Q(A_1B_2C_2) - Q(A_2B_1C_2) - Q(A_2B_2C_1)\\ &+ 2Q(A_2B_2C_2) - Q(A_1B_1) - Q(A_1B_2) - Q(A_2B_1)\\ &- Q(A_2B_2) + Q(A_1C_1) + Q(A_1C_2) + Q(A_2C_1) + Q(A_2C_2)\\ &+ Q(B_1C_1) + Q(B_1C_2) + Q(B_2C_1) + Q(B_2C_2)] \leqslant 1,\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{1}{3}[&- Q(A_1B_1C_1) + Q(A_1B_1C_2) + Q(A_1B_2C_1) + Q(A_2B_1C_1)\\ &- Q(A_2B_2C_2) - Q(A_1B_2) - Q(A_2B_1) - Q(A_2B_2)\\ &- Q(A_1C_2) - Q(A_2C_1) - Q(A_2C_2) - Q(B_1C_2)\\ &- Q(B_2C_1) - Q(B_2C_2) + Q(A_1) + Q(B_1) + Q(C_1)] \leqslant 1,\end{aligned} \tag{21}$$

respectively. By the reformation, the violation degrees of the two inequalities can be compared directly. Comparing the results of the inequality given in Ref. [12], our inequality (18) is indeed more resistant to noise. It seems that we could derive some new three-qubit Bell inequalities, which would be more highly resistance to noise, if we get other Bell inequalities of tripartite d-dimensional ($d>4$) quantum systems.

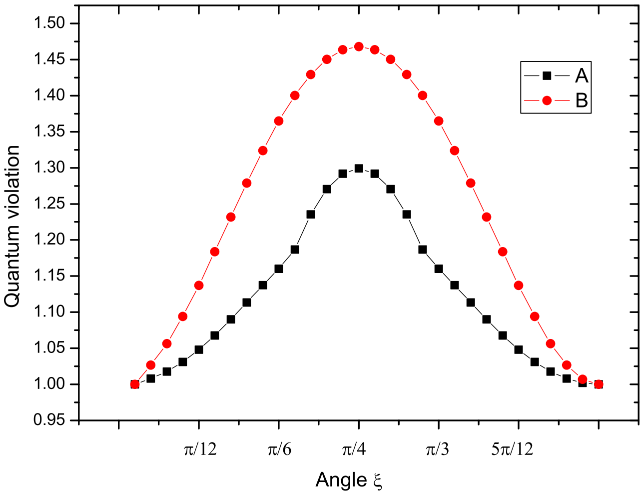

FIG. 3. (Color online) Violation of two Bell inequalities for the generalized GHZ state of three qubits with different value of ξ, where curve A is for the inequality given in Ref. [12] and curve B is for our inequality (18).

When setting $C_1 = -1$, $C_2 = 1$, inequality (18) reduces directly to the two-qubit CHSH inequality

$$E(A_1B_1) - E(A_1B_2) - E(A_2B_1) - E(A_2B_2) \leqslant 2,$$

which is equivalent to

$$E(A_1B_1) + E(A_1B_2) + E(A_2B_1) - E(A_2B_2) \leqslant 2.$$

Starting from the Bell inequality for three five-level systems, another three-qubit Bell inequality can be obtained as

$$\begin{aligned}P(&a_1 + b_1 + c_1 = 1) - 2P(a_1 + b_1 + c_1 = 2)\\ &+ P(a_1 + b_1 + c_1 = 3) + P(a_1 + b_1 + c_2 = 1)\\ &+ P(a_1 + b_1 + c_2 = 2) + P(a_1 + b_2 + c_1 = 1)\\ &+ P(a_1 + b_2 + c_1 = 2) + P(a_2 + b_1 + c_1 = 1)\\ &+ P(a_2 + b_1 + c_1 = 2) + P(a_1 + b_2 + c_2 = 0)\\ &- 2P(a_1 + b_2 + c_2 = 1) + P(a_1 + b_2 + c_2 = 2)\\ &+ P(a_2 + b_1 + c_2 = 0) - 2P(a_2 + b_1 + c_2 = 1)\\ &+ P(a_2 + b_1 + c_2 = 2) + P(a_2 + b_2 + c_1 = 0)\\ &- 2P(a_2 + b_2 + c_1 = 1) + P(a_2 + b_2 + c_1 = 2)\\ &+ P(a_2 + b_2 + c_2 = 0) + P(a_2 + b_2 + c_2 = 1)\\ &- 2P(a_2 + b_2 + c_2 = 3) \leqslant 4.\end{aligned} \tag{22}$$

The above inequality is just the Mermin inequality when it is expressed in terms of correlation functions:

$$- Q_{111} + Q_{122} + Q_{212} + Q_{221} \leqslant 2. \tag{23}$$

IV. SUMMARY

To summarize, we have presented the tight Bell inequalities expressed by probabilities for three four- and five-dimensional systems. The tight structure of Bell inequalities for three d-dimensional systems (qudits) is also proposed. Moreover, a new Bell inequality for three qubits is derived (or, say, reduced) from the inequality for three four-level systems. Inequality (18) is violated by any pure three-qubit entangled state, and it is more resistant to noise compared with the one given in Ref. [12]. Furthermore, the tight Mermin inequality for three qubits can be obtained by reducing the Bell inequality for three five-level systems [see Eqs. (10) and (22)].

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