Concretisations: A Support for Teachers to Carry Out Instructional Innovations in the Mathematics Classroom

Abstract: We recognize that though teachers may participate in various forms of professional development (PD) programmes, learning that they may have gained in the PD may not always lead to corresponding perceivable changes in their classroom teaching. We offer a theoretical re-orientation towards this issue by introducing a construct we term "concretisation". Concretisations are resources developed in PD settings which can be converted into tangible tools for classroom use. In theorising such resources, we contribute in informing the design process of teacher professional development for better impact into actual classroom practice. We purport principles of design which render concretisations effective. Subsequently, we test these principles by presenting a specific case of teaching mathematical problem solving.

Keywords: concretisations, instructional practices, mathematics teacher development, problem solving

Introduction

When teachers participate in teacher professional development (PD) programmes, it is often assumed that they would be able to take something of value from these sessions that will consequently influence their classroom teaching. As PD providers for mathematics teachers, this is certainly a conscious goal for us when we design and implement PD programmes. In our experience, however, we often found that while teachers may appear to participate productively and even avow learning during these PD sessions, there are often little teaching changes corresponding to the avowed learning as observed in classroom enactments. We term this common phenomenon (e.g., Hill, 2009; Wallace, 2009) the "PD-Classroom gap". For instance, in a survey that sought teachers' views about the effectiveness of certain PD opportunities for learning about teaching (Smylie, 1989), district-sponsored in-service programmes were rated least effective out of 14 possibilities while direct teaching experience were rated most effective.

One way to close the PD-Classroom gap is to bring PD work closer to the classroom. This has been attended to in a variety of ways such as focusing on problems of practice during PD and overlapping PD sites with classroom work, such as in Lesson Studies. These are useful enterprises and we have adopted a combination of them in our PD work. In this paper we go beyond suggesting practical approaches. Instead, we present a theoretical treatment that provides parameters that could explain (and hence be applied to address) this PD-Classroom gap. A significant contribution in our approach is the construct of *concretisation*. We begin by positioning concretisation within a conceptual framework that links work done in PD settings to teacher instructional work in the classroom. This is followed by a description of principles underlying the design of concretisation. In the empirical section, we present a particular case of concretisation in the context of supporting mathematics problem solving instruction.

Conceptual Framework

The classroom and the PD domain

We begin conceptualising this PD-Classroom gap by examining the two domains that are relevant in this discourse – the PD setting and the classroom setting. We start with the domain of classroom practice since if the goal of PD is to influence instructional work in the classroom, then one should start the process of PD design with the needs of classroom practice. Arguing along a similar vein, but in her case referring to the gap between reform curriculum and practice, Ball (2000) stated that the way to address the gap should not begin by analysing the new curriculum to prescribe what teachers should know, because “little is known about how ‘knowing’ the topics on these [curricula] lists affects teachers’ capabilities” (p. 244). Rather, "our understanding of the … knowledge needed in teaching *must start with practice*" (p. 244, emphasis added).

In considering instructional work in the classroom, we propose a basic model (see right-hand portion of *Figure 1*) adapted from the well-known instructional triangle (Cohen, Raudenbush, & Ball, 2003; Lampert, 2001) involving "Teacher", "Students", and "Content". However, we narrowed the foci of inquiry from "Teacher" to "Teacher Goals" and "Content" to "Tools" to indicate our more sharpened fields of study about classroom instruction. These refinements are influenced by prevailing research on key levers that can directly influence the quality of teaching.

Our attention to "Teacher Goals" is drawn from Schoenfeld's (2011) Resources-Orientations-Goals model for analysing classroom teaching actions. Resources refer to the mental toolset, such as knowledge and metacognition, that the teacher draws upon to plan and enact teaching. Orientations encompass teacher beliefs about and dispositions towards mathematics as well as the teaching of mathematics. Goals refer to the instructional objectives that teachers identified prior to a lesson and those that emerged only during teaching. To keep to a simple model as sufficient for our purposes, we have chosen to include only goals in the model. This is not unprecedented as Schoenfeld's earlier works were done mainly in goals analysis (e.g., Schoenfeld, Minstrell, & van Zee, 2000).

As reflected in the figure, we conceive of "Teacher Goals" as directly acting on "Students" and "Tools" in the classroom. That is, in classroom teaching, apart from the teachers directly instructing students, their work is also often mediated by tools. By tools, we mean external representations of mathematical content that are accessed or acted upon by the teacher and the students in the classroom. Examples of tools include written work on the whiteboard, tasks presented to students to work on, and curricular materials in the form of textbooks. The potential of these tools to influence the quality of learning in the classroom is attested by the scholarly work done in task design (e.g., Watson & Ohtani, 2015) and the influence of curricular materials in the classroom (e.g., Remillard, Herbel-Eisenmann, & Lloyd, 2000).

Similar to the basic model of classroom practice, we propose a model of interactions within the PD setting which involves the triad of “Teacher Educator Goals”, “Teacher Goals”, and “Concretisations” (see left-hand portion of *Figure 1*). This is similar to Nipper and Sztajn’s (2008) extension of the instructional triangle to PD. While this by no means present all the elements involved in PD, it captures much of the work we do during PD sessions with teachers. The reflected interplay between “Teacher Educator Goals” and “Teacher Goals” represents how we interact with teachers with a view of influencing the goals that they would bring into classroom practice. As teacher educators we take responsibility of teacher learning during PD sessions. However, our interactions are negotiations about instructional goals conceived by both teacher educators and teachers as reflected in the bidirectional arrow. This explicit articulation and clarification of goals within the PD setting help in teacher buy-in and strengthens teachers’ grasp of how instructional innovations developed during PD can be carried out in the classroom.

We think that it is not sufficient to discuss conceptualisations of teaching innovations during PD and leave teachers to work out the specifics of implementations in the classroom. To encapsulate the negotiated goals of instruction, there is a need to translate it into a form that is directly useable in the classroom to be acted upon by teachers and engaged with by students. We call this concrete objectification “concretisation”. This concretisation is then utilised as a significant tool in the classroom to advance the teacher’s goals, including the goals to support the instructional innovation.

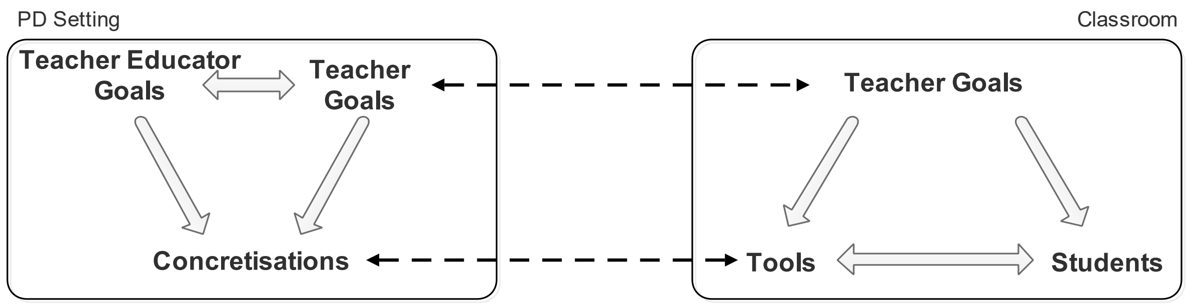

Figure 1. A model that illustrates the linked contexts between the PD setting and the classroom.

The PD-Classroom Gap

The work in the PD setting is meant to relate closely to the teachers' work in the classroom. Hence the expectation that classroom settings will be influenced by whatever learning has taken place in PD setting. However, this influence may fail to materialise which suggests that there exists what we have referred to as a PD-Classroom gap. To aid our conceptualisation of PD work that can effectively support teachers' work of carrying out instructional innovations in the classroom, we focus on those portions that overlap between the PD setting and the classroom setting. In this respect, we were influenced by Engle's (2006) work on *intercontextuality*: "Intercontextuality occurs when two or more contexts become linked with one another. When this occurs between learning and transfer contexts, the content established during learning is considered relevant to the transfer context" (p. 456). In our model, there are two such linking of contexts, as represented by the two perforated arrows in *Figure 1*: (1) "Teacher Goals" within the PD context of designing for instructional innovation is directly linked to "Teacher Goals" within the classroom context of carrying out the instructional design; (2) "Concretisations" crafted within the PD setting is specifically meant to be useable as "Tools" in the classroom. In this paper, our focus is on the second linkage involving concretisations. The first linkage related to goals is important as well and we attend to strengthening this linkage through closely working with teachers in our PD. However, we consider the second linkage to be more crucial because working on concretisations in the PD settings is also made with a view of strengthening teachers' goal alignments. (This aspect is made clearer under the section 'Concretisations'.)

Concretisations

We clarify that concretisations are tangible expressions of the goals and ideals of instructional innovations that teacher educators and teachers agree upon in the PD setting to realise in the classroom. They can thus be regarded as resources that engage teachers in learning within PD settings that can be converted into tools for classroom use. Resources that lack this critical attribute of useability in the classroom as a concrete tool are not considered concretisations. For example, a mathematical task sheet that teacher educators and teachers work through during PD is a concretisation if the same sheet (possibly with modifications) can be employed by the teachers in their classroom teaching. On the other hand, consider PD resources such as authentic students' work and video clips of classroom teaching. These resources can aid teachers to reflect on students' thinking with a view of altering instructional work and we should add that these were employed in our PD work with teachers as well. However, teachers cannot use these artefacts readily in teaching, hence, they are not concretisations.

Concretisations are meant to support teachers' efforts at adopting innovative practices. As such, it is important that they are carefully designed. We purport these design principles underlying the construction of concretisations: (1) comportment, (2) integration, (3) accessibility, and (4) visibility.

Comportment. By comportment we mean the extent to which the concretisation 'carries' the joint goals of the teacher educators and the teachers, including the goals that support the instructional innovations. Thus, the design for comportment takes place primarily within the PD setting as it is the site where the goal discussions among the teacher educators and the teachers take place. However, it is not uncommon to observe resources that were designed for certain instructional aims but underwent “lethal mutations” (Brown & Campione, 1996) when used in the classroom. So, considerations of comportment should ‘feedforward’ in a sense of ‘carrying’ the goals into the classroom. As such, concretisation are to be imbued with tight links between the intended classroom activity and the underlying principles that drive the innovation, so that the performance of work required by the activity—by both teachers and students—would raise the likelihood of invoking desirable principles involved behind the design of the tasks. In other words, if comportment is truly adhered to in the design, it is not possible for the concretisation to be used in the classroom without the participants tapping into the goals it comports.

Integration. Integration refers to the smooth weaving of the concretisation into the work of teachers’ regular instructional practices in the classroom. Concretisations should be designed with an eye for teachers’ ease of use in the classroom. In particular, the use of the concretisation should not consume so much cognitive load (c.f., Sweller, 1988) for the teacher during the work of teaching that it affects attention to other instructional essentials in the classroom. One way to reduce cognitive load for the teachers is to make concretisations cohere with the routines employed by the teachers in the classroom. Concretisations that require a teacher to depart too far from the norms of classroom work can render them as cumbersome add-ons instead of effective tools to support innovative practices.

Accessibility. This principle relates both to the students’ ease of use of the concretisation for learning and their access to substantial mathematical content in the process. While concretisations should be easily integrated into teachers’ routines, they should not be mere demonstration tools for teacher talk but also tools that students work on for learning. For concretisations to be able to promote student learning, students in the process of working with the concretisations should be led to make sense of the mathematical content integrated in its design. In other words, concretisations present themselves as conduits for students to engage in the mathematics promoted by the teaching innovation; they allow students who may have different points or level of entry to access the mathematics; and there are built-in features to engage the students in learning without reducing the overall cognitive demand. This quality of cognitive demand of the mathematical activity is akin to the cognitive demand dimension in Schoenfeld's (2013) TRU Math (teaching for robust understanding of mathematics) framework for classroom observation. On one end of this dimension, activities may reduce the mathematical challenge to a set of procedures rendering the students' engagement with the mathematics trivial. On the other end, activities engage students in productive struggle with the mathematics. Concretisations are envisioned to enable activities to fall within the latter end.

Visibility. There are two aspects to our conception of visibility. First, the concretisation should be in a form where students can make visible connection to easily. In other words, the form in which the teachers work on should correspond closely with the form that students work with directly. The second level of our conception of visibility relates to how the concretisation renders the mathematics visible (Ritchhart, Church, & Morrison, 2011). We take the perspective that mathematics involves a process of thoughtful reasoning that may not always proceed in a straightforward manner. The use of concretisations should make this reasoning process visible by providing a means of capturing and monitoring it. It thus provides teachers and students with a common platform to study the underlying processes in doing mathematics.

Our conceptual framework links the work we do in the PD setting to the classroom with respect to supporting teachers in carrying out instructional innovations. We propose the use of concretisation in strengthening intercontextuality to address the PD-Classroom gap. We claim that concretisations, when crafted according to the explicated design principles, can advance the goals that support the teaching innovations incubated in the PD setting. Our conceptualisation so far is based on teaching mathematics generically. In the next section, we zoom in to the context of our specific study: the innovation in the form of teaching mathematical problem solving.

Context of the Study

The Innovation: Mathematical Problem Solving (MPS) Instruction

MPS has been identified as "central to mathematics learning" (Ministry of Education, 2006) in the Singapore mathematics curriculum since the 1990s. However, similar to the situation internationally (e.g., Doorman et al., 2007; Schoenfeld, 2007), the actual use of MPS in the teaching of mathematics remains largely elusive for various reasons. Teachers face a number of difficulties in enacting problem solving in the classroom. Most teachers are unable to effectively teach problem solving because of their personal lack of knowledge and expertise in problem solving (Foong, Yap, & Koay, 1996). The predominantly expository teaching approach used by Singapore teachers (see for example Chang, Kaur, Koay, & Lee, 2001; Yeo & Zhu, 2005) also does not provide a conducive setting for problem solving instruction (Fan & Zhu, 2007). The examination culture (Kaur & Yap, 1998) in Singapore classrooms further reinforces the practice of expository teaching of examinable content and hinders innovations that require coverage of content outside the scope of examinations. Finally, teachers are not likely to have a clear image of what successful teaching of problem solving in actual classrooms would look like.

We thus recognised that any design to facilitate teacher competence for MPS instruction would have to attend to the re-orientation of teachers' goals towards problem solving and the expansion of their repertoire of practice. In view of this, when we embarked on a project that sought to render the teaching of MPS more ubiquitous in Singapore mathematics classrooms teacher development was a key component. For more details about the design and implementation of the PD, the outline and structure of the lessons, problems used, how some of the teachers' MPS lessons developed, as well as how some students have taken to the instructional innovation, readers may refer to some of our previous publications (e.g., Leong, Dindyal, et al., 2011; Leong et al., 2014; Leong, Tay, Toh, Quek, & Dindyal, 2011; Leong, Toh, Tay, Quek, & Dindyal, 2012; Toh, Quek, Leong, Dindyal, & Tay, 2011). In this paper, we focus only on how teachers and students have engaged in a model for MPS in the context of implementing problem solving instruction in the classroom.

Pólya's Model for MPS

We chose the well-established Pólya's (1945) four-stage model for MPS. A mainstay in the Singapore mathematics curriculum documents, but has not been commonly utilised in the classroom, we thought that it was likely to be met with less resistance by teachers. The first three stages – Understanding the Problem (UP), Devising a Plan (DP), and Carrying out a Plan (CP) – refer to consciously considering the situation presented in the problem and laying out a plan (or plans) for approaching the problem based on this consideration, before finally carrying out the planned approach. The fourth stage of "Looking Back", which we renamed as Check and Expand (CE) to reflect better what it refers to, includes making checks on whether the solution obtained is correct, expanding on the given problem possibly by looking for new approaches to solve the problem, and making new problems related to the original one. It is important to emphasize that the stages are cyclic hence a solver can "loop back" to previous stages. For example, suppose a solver at the third stage of CP suddenly realised that he made a mistake in UP. The model directs the solver to go back to the first stage of the process.

We worked closely with teachers in PD settings to develop materials for teaching MPS using Pólya's model. Our engagement with teachers followed the mode laid out in *Figure 1* and elaborated in the corresponding section. It became clear to us that although most teachers had cursory familiarity of Pólya's model, they were unsure of how the model can help in their own and their students' MPS. That prompted the initiative to design a tool of engagement with Pólya's model that would support the discussion with teachers within the PD setting as well as be useable for teaching later in the classroom—that is, a concretisation of Pólya's model.

The Concretisation of Pólya's Model

In keeping with the design principles we listed for concretisations, there were two components to the concretisation: the Practical Worksheet and the Four-Panel Presentation.

The Practical Worksheet (PW). The Practical Worksheet (PW) is a four-page worksheet framed according to Pólya's four stages (for more details, refer to Toh et al., 2011). In particular, each page corresponds to a section labelled as Understand the Problem, Devise a Plan, Carry out the Plan, and Check and Expand respectively. Intended to guide solvers in their use of Pólya's stages for MPS, each section includes questions that solvers reflect on and answer as they go through the different stages. It also encourages solvers to loop back when the situation calls for it. The text that appears in the original worksheet in relation to Pólya's stages is provided in *Figure 2*. It must be noted, however, that over the duration of the project, the format of and text in the PW have been modified in various ways according to how teachers thought their students would be able to engage in the MPS process using Pólya's model more effectively.

In matching the characteristics of the PW with the design principles of concretisations, the worksheet fulfils some of the characteristics. First, the PW comports with the instructional goal of highlighting the use of Pólya's model in MPS. In addition, during PD, teachers made use of the PW in their own problem solving activities enabling them to gain familiarity with Pólya's

I. Understand the problem
(You may return to this section a few times. Number each attempt to understand the problem accordingly as Attempt 1, Attempt 2, etc.)

(a) Write down your feelings about the problem. Does it bore you? Scare you? Challenge you?
(b) Write down the parts you do not understand now or that you misunderstood in your previous attempt.
(c) Write down your attempt to understand the problem; and state the heuristics you used.

II. Devise a plan
(You may return to this section a few times. Number each new plan accordingly as Plan 1, Plan 2, etc.)

(a) Write down the key concepts that might be involved in solving the problem.
(b) Do you think you have the required resources to implement the plan?
(c) Write out each plan concisely and clearly.

III. Carry out the plan
(You may have to return to this section a few times. Number each implementation accordingly as Plan 1, Plan 2, etc., or even Plan 1.1, Plan 1.2, etc., if there are two or more attempts using Plan 1.)

(i) Write down in the *Control* column, the key points where you make a decision or observation, for e.g., go back to check, try something else, look for resources, or totally abandon the plan.
(ii) Write out each implementation in detail under the *Detailed Mathematical Steps* column.

IV. Check and Expand

(a) Write down how you checked your solution.
(b) Write down your level of satisfaction with your solution. Write down a sketch of any alternative solution(s) that you can think of.
(c) Give one or two adaptations, extensions or generalisation of the problem. Explain succinctly whether your solution structure will work on them.

Figure 2. The text related to Pólya's stages found in the original PW in Toh et al. (2011).

model. This strengthens comportment with respect to building links for teachers between the classroom activity and the underlying problem solving model. Second, in line with the principle of integration, it is common for Singapore mathematics teachers to give out tasks to students in the form of worksheets. While the MPS tasks given in the PW differ from the mathematical exercises students do for seatwork, this classroom norm of students working on tasks while teachers walk from table to table to over their progress is an easy integration into the routines of teachers' classroom practices. Third, the PW is designed to be user-friendly providing accessibility for students to use Pólya's model for MPS. In particular, the worksheet contains scaffolds via the labelling of Pólya's stages and the guide questions to help students monitor their MPS process. On visibility, however, the worksheet was not sufficient in itself to serve as a common visual platform between teacher and students to study the MPS process. This leads us to the next component of the concretisation.

The Four-Panel Presentation (4PP). The Four-Panel Presentation (4PP) is a manner of utilising the whiteboard for modelling the problem solving process using Pólya's stages. In particular, the whiteboard is divided into four sections, allotting a section for each of the stages (see *Figure 3*). The intent is then to use the panels to model the thought process involved in solving a problem using Pólya's stages by moving from panel to panel, and sometimes backwards to illustrate any relevant looping in the process.

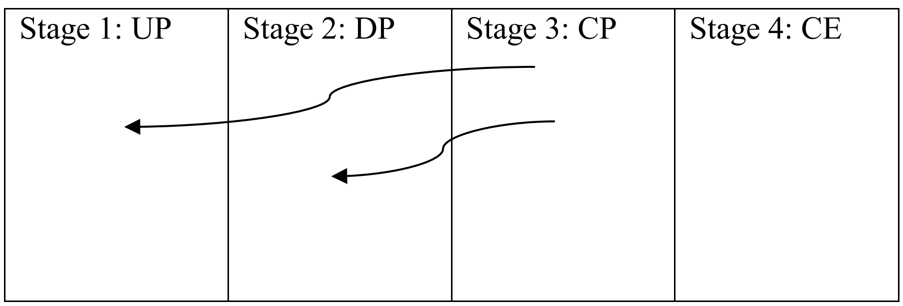

Figure 3. The use of the whiteboard in teaching problem solving in a 4PP.

The 4PP was first observed in one of the teacher's lessons. Since then, we have used it as a component of the concretisation of Pólya's model in succeeding PD sessions with all teachers. In particular, during the sessions we would use the 4PP to develop our discussions of the problems that were part of the MPS lessons. This also served to model for teachers how they can use the 4PP during lessons. Our adoption of the 4PP in the PD grew out of recognising its potential to fulfil especially the "visibility" principle of design which was an aspect lacking in the PW alone. The 4PP allows teachers to make visible the problem solving process along the lines of Pólya's stages. It also allows students to refer to a visual representation for comparison to the work they do on the PW. Thus, we encouraged teachers to use the 4PP. A snapshot of how a teacher made use of the 4PP in the classroom is given in the Supplementary Material.

The 4PP strengthens the design principle of comportment in that it is also designed to highlight the use of Pólya's model for MPS like the PW. It also conforms to the principle of integration since the use of the whiteboard is common in Singapore classrooms. The 4PP, however, does not easily align with the principle of accessibility because typically it is the teachers who interact with the 4PP and not the students. But taken together, the PW and the 4PP form a concretisation of Pólya's model that possesses all the design principles previously enumerated. Following from our theorisation of a concretisation, the PW and the 4PP should then support the teaching of Pólya's model to help students learn about MPS.

We now turn to the empirical section of this paper. As elaborated earlier, our interest is in addressing the PD-Classroom gap or the gap between teacher learning experiences in PD settings and useable instructional practices in the classrooms. In particular, we examine the potential of concretisations (designed in PD settings) in supporting the goal of teaching and learning Pólya's model (in the classrooms). We study the potential along the lines of the design principles of concretisations. The first principle of comportment relates to how tightly Pólya's model is worked into the concretisations. This is mainly the joint work of the teacher educators and teachers within the PD setting and is thus outside the scope of our current inquiry. Moreover, the reader can make a judgment on comportment of the PW and 4PP based on our elaborations of their design. In particular, one can assess the validity of our claim that participants who use the PW and 4PP cannot do so without also attending to the intended underlying Pólya's processes.

Our inquiry here zooms in on the 'movement' from the PD setting to the classroom in which the principles of integration, visibility, and accessibility are in play. The following research questions correspond to these principles:

1. How do teachers *integrate* the 4PP developed during the PD sessions in their classrooms?

2. How do teachers utilise the 4PP to facilitate *visibility* of the MPS?
3. Did the PW provide students *access* to engage in MPS guided by Pólya's stages?

Method

Participating Schools

We carried out our study in two secondary schools in Singapore – Westpark[1] Secondary School and Eastpark Secondary School during the 2013 school year. Both schools have been involved in the project since 2012. As part of the project, the mathematics teachers of the Secondary 1 students (13-14 year olds) in each school conducted a 10-lesson MPS module with their students over a 10-week period. The module culminated with administering a final MPS test for students at the end of the 10 weeks. Students took the test individually and they were given one-hour to complete it in class.

Data Collection

At the time of the study, there were four mathematics teachers in each school teaching the module for different classes where the average class size was 40. Due to limited resources, we focused more attention on the classroom implementation of the MPS module carried out by the teachers who were tasked to take the lead in the implementation – Leonard for Eastpark and Megan for Westpark. All their ten lessons for the MPS module were videotaped for the study. For the other teachers, their lessons were videotaped between two to four times over the period of the study. For all the teachers only one video recorder was used. It was situated at the back of the classroom and the view was zoomed out to capture a broad view of the classroom. It was also horizontally panned whenever necessary to keep the teacher always in view of the screen.

To address the question related to students' access to the MPS process using the concretisation, copies of the students' PWs of the final MPS test were obtained for analysis. We were unable to collect the PWs of the students from Eastpark Secondary School during the same school year as when the teacher data were collected. Thus, the data used for this component of the study only consisted of the 158 PWs from four classes in Westpark Secondary School.

Data Analysis

We reviewed the video recordings of all the teachers' lessons. Typically, an MPS lesson would have a portion for homework (HW) as well as for what was called the problem-of-the-day (POD). For each portion, we noted whether and how the teacher carried out the 4PP by indicating if a certain stage of Pólya's model was reflected in their in-class board work, and whether the process of looping was carried out as well.

To capture nuances in how teachers integrated the 4PP and made use of it to facilitate visibility, we analysed Megan and Leonard's videos. We examined how they used the 4PP to reinforce the problem solving processes and the use of the PW. We paid attention to their different verbal and non-verbal cues during their whole class instruction and their table-to-table interaction with students. We transcribed their utterances and described the actions they made.

For the students' work on their final MPS test, we reviewed and analysed them according to whether the students carried out Pólya's model appropriately as reflected by what they wrote in their PWs. We coded students' works for each of Pólya's stage according to whether it cohered (CY) or did not cohere (CN) with the intended use of the stage, or if it was blank or vacuous (BV). We also determined if the students reflected a process of looping (LY) in their worksheets. Two researchers were involved during coding. The researchers independently coded 25 worksheets. Upon comparing, only one item was found to be different. A consensus was reached on the final code after discussion. One of the researchers coded the remaining 133 student scripts. For a better appreciation of the coding process, we provide the test problem and some excerpts of students' works that were coded in a particular way in the Supplementary Material.

Findings

In this section, we first present tables that summarise the teachers' use of the 4PP. These provide a basis for the next portion where we share the findings on how teachers integrate the 4PP in their instruction and utilise it for visibility of Pólya's model and MPS. The last part of this section then presents the findings on students' use of Pólya's model for MPS.

The Teachers' Use of the 4PP over 10 Lessons

Table 1 and Table 2 provide a summary of the use of the 4PP in Megan and Leonard's class respectively. In the interest of space, in Table 3 we summarise the same for only two teachers – Lara and Cathy from Eastpark and Westpark Secondary School respectively. Data obtained for the other teachers can be found in the Supplementary Material. The tables display information on whether teachers utilised the 4PP in discussing a particular HW or POD in class for each of Pólya's stages. The tables also show if the process of looping was reflected in the 4PP during the discussion for each problem. A check mark (√) on the table indicates that the segment of instruction was carried out while utilising the 4PP. An x mark (x) indicates that the segment of instruction was carried out but the 4PP was not utilised. Finally, "NA" means that the segment of instruction was not carried out at all during the lesson, hence the use of the 4PP was not applicable.

Teachers' Integration of the 4PP in their MPS Lessons

Table 1 and Table 2 show that Megan and Leonard used the 4PP for POD and HW on a regular basis during the teaching of the MPS module. Furthermore, when they did not use the 4PP it seemed to be deliberate so as to fit an instructional plan. In particular, both Megan and

Leonard did not use the 4PP during the first POD and HW because Pólya's model was not yet introduced during these times. In the other instances that Megan did not use the 4PP (i.e., Lesson 4, 5, and 10), the discussion was not framed within a 4PP because she was building on solutions that her students presented. Hence, it appeared that the lead teachers found the 4PP easy to integrate in their instruction as reflected by their regular and deliberate use.

Table 1

Summary of Megan's use of the 4PP over the ten lessons

Lesson No	Item	Pólya's Stage				Looping	Remarks
		UP	DP	CP	CE		
1	HW	NA	NA	NA	NA	NA	First lesson. No homework to work on.
	POD	x	x	x	x	x	
2	HW	x	x	x	x	x	Pólya's stages introduced.
	POD	√	√	√	x	√	
3	HW	√	√	√	NA	x	
	POD	√	√	√	NA	x	
4	HW	NA	NA	NA	NA	NA	Teacher built on student's solution, hence no 4PP.
	POD	x	x	x	x	√	
5	HW	x	x	x	x	x	Student presented HW. PW introduced. Did not finish last stage due to time constraint.
	POD	√	√	√	NA	√	
6	HW	√	√	√	√	√	Stage 4 (CE) was introduced. POD stage 4 left for homework.
	POD	√	√	√	NA	√	
7	HW	NA	NA	NA	x	x	Homework only looked at stage 4.
	POD	√	√	√	√	√	
8	HW	√	√	√	√	√	
	POD	√	√	√	√	x	
9	HW	√	√	√	√	√	
	POD	√	√	√	√	x	
10	HW	x	x	x	x	x	Student presented homework.
	POD	√	√	√	√	√	

√ – reflected in 4PP x – not reflected in 4PP NA – Not applicable

Table 2

Summary of Leonard's use of the 4PP over the ten lessons

Lesson No	Item	Pólya's Stages UP	DP	CP	CE	Looping	Remarks
1	HW	NA	NA	NA	NA	NA	First lesson. No homework to work on.
	POD	x	x	x	x	x	
2	HW	x	x	x	x	x	Pólya's stages introduced.
	POD	√	√	√	√	x	
3	HW	√	√	√	√	√	Only completed stage 1 of POD due to time constraint
	POD	√	NA	NA	NA	NA	
4	HW	√	√	√	√	√	
	POD	√	√	√	√	x	
5	HW	√	√	√	√	√	Did not complete stage 4 of POD due to time constraint
	POD	√	√	√	NA	x	
6	HW	√	√	√	√	x	Explained stage 4 (CE).
	POD	NA	NA	NA	NA	NA	
7	HW	NA	NA	NA	NA	NA	Focused on stage 4. Did not discuss POD.
	POD	NA	NA	NA	NA	NA	
8	HW	√	√	√	NA	√	Stage 4 of homework was not discussed. Focused on the discussion about 'proof'.
	POD	NA	NA	NA	NA	NA	
9	HW	√	√	√	√	√	Lesson ended at CP, CE was meant for homework
	POD	√	√	√	NA	x	
10	HW	√	√	√	√	x	Revision
	POD	NA	NA	NA	NA	NA	

√ – reflected in 4PP x – not reflected in 4PP NA – Not applicable

We recognize that the number of lessons we videotaped for the other teachers is limited and may not provide a full picture of the other teachers' instruction of the modules. Nevertheless, based on the data that we have, except one teacher from Eastpark Secondary School, all the other teachers were observed to have used the 4PP at least once in their classroom. This provides us with better confidence that the 4PP is a feasible and acceptable practice for use in teaching MPS. However, it will be observed from Table 3 that Lara and Cathy did not always use the 4PP to

Table 3

Summary of the two other teachers' use the 4PP for certain lessons in Eastpark (E) and Westpark (W) Secondary School

Lesson No.	Teacher (School)	Item	Pólya' s Stages				Looping	Remarks
			UP	DP	CP	CE		
1	Lara (E)	HW	NA	NA	NA	NA	NA	
		POD	x	x	x	x	x	
4	Lara (E)	HW	x	x	x	x	x	Focused on heuristics. Did not proceed to CE.
		POD	√	√	√	NA	x	
5	Cathy (W)	HW	x	x	x	x	x	PW was introduced.
		POD	√	√	√	√	x	
6	Lara (E)	HW	√	√	√	√	√	Focused on explaining rubric and teaching stage 4.
		POD	NA	NA	NA	NA	NA	
	Cathy (W)	HW	NA	NA	NA	NA	NA	Rubric and teaching of CE introduced.
		POD	√	√	√	√	x	
10	Lara (E)	HW	x	x	x	x	x	
		POD	x	x	x	x	x	
	Cathy (W)	HW	x	x	x	x	x	Did not complete the problem due to time constraint.
		POD	x	√	√	NA	x	

√ – reflected in four-panel presentation x – not reflected in 4PP NA – Not applicable

discuss the rest of the problems in the module. The rest of the teachers also exhibited this inconsistent use of the 4PP. This indicates that not all the teachers may have integrated the use into their regular practice. When not using the 4PP, verbal discussion of the stages and the use of presentation slides were adopted instead. When necessary, they used the whiteboard when discussing certain stages without using the full 4PP spread.

When we zoomed in on how Megan and Leonard utilised the 4PP, we identified similarities and also differences. Both Megan and Leonard used the 4PP with an intention to produce a whiteboard version of Pólya's model for joint reference between teacher and students. Megan and Leonard manifested this by their deliberate act of partitioning the whiteboard into panels and labelling them so that they would correspond to the four stages in Pólya's model. As they went over the problem during the whole class discussion, they moved progressively from panel to panel, or moved back to previous panels for looping when necessary, to demonstrate how problem solving using Pólya's model could ensue. That is, close to the pattern illustrated in *Figure 3*. When workings on a certain stage were referred to, both teachers walked towards the corresponding panel and point or place a hand on it to refer to the workings written there. There was also careful labelling of subsequent attempts or plans for solving the problem.

However, the manner by which Megan and Leonard produced the 4PPs differed. Megan treated it as a reference for students to follow. That is, Megan seemed to have a fixed image of how the 4PP should look like at the end of the session. She took charge of putting up model answers on the panels during both the HW and POD discussions. She identified the key words from the problem and the heuristics that would be used. She would also lead the class in carrying out the proposed solution and expansions. When she elicited students' responses, it was mainly for the purpose of having the students complete a thought that she started. As a result, Megan would have a complete and well-structured 4PP at the end of the discussion of the problem.

On the other hand, Leonard treated the 4PP as a space to build students' MPS attempts as reflected by how he often involved the students in developing the panels. For example, in discussing the HW, Leonard at one time asked a student to share her answer by making her write her solutions on a four-panel set-up on the board. Leonard then made corrections on how she carried this out; in the process, the proper use of the PW was reinforced. For the POD discussion, he would take time to draw out answers from the students first. For instance, he would ask them how they felt about the problem, or what heuristics they planned to use for solving it. Leonard would then write their responses on the respective panels and build on these during further discussion. In the event that his planned agenda will not surface, he would make a suggestion and ask the class to consider it.

To provide readers with a better idea of how Megan and Leonard contrasted in their integration of the 4PP in their lessons, *Figure 4* juxtaposes transcripts of excerpts from Megan and Leonard's classroom instruction as they were both developing the panel for Devise a Plan for a certain problem. In Megan's transcript, notice how she asked students to name or complete the ideas that she had already suggested ("What heuristics am I using?" "So what is the key concept here? The idea of average with what?"). On the other hand, Leonard solicited ideas from the students to fill up the panel ("Can someone suggest?"). And when a student suggested a feasible heuristic, he then encouraged the class to pursue it ("Maybe can. How?").

Despite Megan and Leonard's differing practice, they both were able to promote Polya's model for MPS in their classrooms. This is evidenced by the fact that they laid out elaborations of the problem solutions for each of Pólya's stages as they used the 4PP even if their manner of doing so was different. Thus, it appeared that the 4PP as a concretisation allowed for flexibility of use while keeping the goals of MPS instruction. From the perspective of "integration", it may be said that the teachers integrated the 4PP according to their preferred instructional approach – Megan with a more teacher-controlled stance, Leonard with a disposition to draw from students' initial responses.

Teachers' Use of the 4PP to Facilitate Visibility

As we described earlier, we conceive of two aspects of visibility. Specific to the context of this study, the first is in terms of providing students with a visual of how Pólya's model can be used to facilitate MPS; the second is in terms of making the process of MPS visible to students.

With regards to the first, the regular use of the 4PP alongside the teachers' support of students' use of the PW reflect the teachers' efforts to make Pólya's model a more tangible MPS tool for

Megan's class: An excerpt from Lesson 8	**Leonard's class: An excerpt from Lesson 5**
MEGAN: So what am I doing here? (*She draws a horizontal line across the first panel to separate her workings and the space*.) What heuristics am I using? Huh? What heuristics am I using? STUDENTS: Write a simpler question. MEGAN: OK, use a suitable, use suitable numbers, instead of algebra first. (*She writes on the second panel as she says the words*.) You have to think of scenarios. So what is the key concept here? (*She writes "key concept" and the other words, as she says them, on the second panel*.) The idea of average with what? Also, I must know the total score … STUDENTS: Total number of games. MEGAN: … and total games or matches. Agree? … … The average comes from the total score, it comes from the total number of games. (*She places her hand on the second panel*.)	LEONARD: There is another way of doing this. Can someone suggest? Let me just read through the heuristics you tell me possible or not. Act it out. STUDENTS: No. LEONARD: Draw a diagram. STUDENTS: No. LEONARD: Hey, "draw a diagram" might work you know. If you are drawing your model, it might work. How about "guess and check"? STUDENTS: No. LEONARD: Use equation or algebra. STUDENTS: Maybe can. LEONARD: Maybe can. How? … … So what if we can use another method? Attempt 3. STUDENTS: Algebra? LEONARD: We use algebra. (*He draws a horizontal line across the second panel and writes "Attempt 3" and "Use algebra"*.) STUDENT-P: That means assume Peter's salary as "*a*" lor. LEONARD: OK, so instead of writing 100%, we write it as? STUDENT-Q: "*a*". LEONARD: As "*a*". … …

Figure 4. Transcripts of an excerpt from Megan and Leonard's classroom instruction when they were both developing the panel for Devise a Plan for a certain problem.

students. As can be seen in the tables in the previous section, teachers used the 4PP for POD and HW discussions substantially. It is not that surprising to us that 4PP was used for the PODs as these were problems deliberately designed to help teachers foreground Pólya's model. More surprising was the regular use of 4PP in presenting the HW solutions. We expected the prevailing culture of HW discussion to be one that focusses mainly on the key strategies and solutions to problems. That the teachers took time to work the steps through the panels reflected a likely deliberate commitment to reinforce the students' use of Pólya's model for MPS. During their table-to-table interaction, the teachers were also observed to further reinforce the discipline of Pólya's model by checking and reminding students to fill in the PW accordingly, referring to how the workings were carried out in the 4PP.

As to the second, an important aspect of MPS that we think is important for students to recognise is how MPS can be "messy" in that it may involve zig-zagging between conjectures and refutations, and exploring alternative strategies. In Pólya's stages, this is embodied in the loop back processes anticipated when solving problems. These loops exhibit how the stages do not always proceed in a linear fashion. Hence, the teachers' use of the 4PP to demonstrate looping, though not consistently, reflected their effort of using the concretisations to make mathematical thinking more visible to students. From Table 1, it can be seen that Megan demonstrated looping back on occasion both for HW and POD problems. On the other hand, as seen in Table 2, Leonard demonstrated looping back on five HW problems. For the other six teachers, three were observed to performing looping back at least once.

Specific only to Leonard, we observed that he was explicit in communicating to his students how the PW should be a record of their thought processes by demonstrating it through his 4PP workings. For example, he would show and stress how unwanted entries were not to be erased but rather only cancelled. Hence, this validated how MPS can be messy and does not necessarily take place in a straightforward manner.

Students' Engagement in MPS Using Pólya's Stages

Table 4 summarises the results of the analysis of the final MPS tests of the students from Westpark Secondary School. It reveals that most of the students were able to use the PW according to its intent based on how they filled up the different components of the worksheet. Some students also carried out looping. It might be argued that the students were simply writing their workings on the designated parts. However, their conscious decision to correspondingly fit the various parts of their working into the PW implies a degree of familiarity with Pólya's model.

Table 4

Results of the analysis of the final MPS tests of the students from Westpark Secondary School

Code	**Pólya's Stages**				**Looping**
	UP	**DP**	**CP**	**CE**	**(LY)**
BV (Blank or Vacuous)	2	2	1	1	-
CY (Cohered with intent)	152	145	157	157	-
CN (Did not cohere with intent)	4	11	0	0	-
LY (Looping present)	-	-	-	-	36

We noted that in Table 4 Pólya's second stage (DP) had the most number of instances where students did not use a stage according to its intent followed by the first stage (UP). This confirmed our observation that students tended to not take the time to determine if they have understood the problem properly, let alone think about a plan for solving the problem. Rather, the natural tendency was to impulsively work on solving the problem straightaway. The working for the problem's solutions would sometimes find itself under the first two stages. But performing the proper steps for the CP stage was unproblematic.

The numbers under the CE column were surprising in an encouraging way. Students do not naturally carry out checking and expanding. As such, we expected that there would be more workings under this stage coded as BV. The fact that this was not the case indicated that students may have developed the habit to carry out this important part of problem solving.

Overall, the findings indicate that the PW allowed students to engage in and hence access MPS using Pólya's model. We conjecture that teachers' use of the 4PP alongside the PW strengthened students' accessibility to the MPS processes tied to these concretisations.

Discussion

In this study, concretisations were designed with these principles: comportment, integration, visibility, and accessibility. We developed concretisations utilising these principles within the context of supporting the teaching and learning of Pólya's model within an MPS module. The findings revealed the following: (1) teachers were able to integrate the 4PP to their instructional practices in ways that were aligned to their pedagogical preferences; (2) teachers used the 4PP to make visible not only the 4 stages of Pólya but also the process of MPS, including loopback; and (3) students demonstrated the ability to access the MPS procedure through Pólya's model as evidenced by the work they did on the PW. Although we did not include comportment in the empirical section of the inquiry, it was a feature that was built in the 4PP and PW.

In view of reports on how difficult it is to enact MPS at scale and how "elusive" actual MPS in classrooms is (Leong et al., 2011; Stacey, 2005), the empirical data showing successful implementation of an MPS module provided in this paper is encouraging. We think that this supports the claim we made earlier: "concretisations, when crafted according to the explicated design principles, can advance the goals that support the teaching innovations incubated in the PD setting."

While the report in this paper focuses on how the concretisations of 4PP and PW that we designed in PD settings were used in the classroom, the actual process is symbiotic (as reflected by the bidirectional arrow joining "concretisation" and "tools" in *Figure 1*). As an example, we continue to make changes to the form of the PW to make it even more accessible to students. In keeping with design experiments, the refinement process is iterative and it 'bounces' between observations of the tool's effectiveness in classroom use and tweaks made during discussions in PD settings. This highlights another critical feature of PD work that is less reported: the need to follow-through on concretisations in terms of refinements to consider classroom realities.

Broadening from the implement of PW and 4PP as specific cases of concretisations within the context of MPS, we think there is potential in advancing the theory surrounding our conception of concretisation as one that strengthens intercontextuality between the PD setting and the classrooms.

The usefulness of concretisation as a construct is particularly pertinent for mathematics teacher educators in their theorising of effective PD – effectiveness in the sense of its impact in actual classroom instruction. In the current literature on PD, there is no lack of lists of criteria for effective PD (e.g., Loucks-Horsley, Stiles, Mundry, Love, & Hewson, 2010; Timperley, 2008). There is, however, little scholarly work done in explaining how each of the items in the respective lists relates to one another within a more comprehensive theoretical scheme. In conceiving concretisation as an objectification of the joint-goals of teacher educators and teachers, we contribute a perspective to theorising effective PD that is not isolationist. Rather, it is located within a theoretical conception that binds together critical elements surrounding PD work: teacher educator goals, teacher goals, tools in the classroom, students, intercontexuality between PD setting and classroom work (see *Figure 1*). To us, concretisation plays a significant role in the converging and clarification of teacher-educator/teacher goals as well as in 'gap-closing' between teacher learning in PD setting and teacher enactment in the classroom.

This theory of concretisation has explanatory power – it accounts for why certain classes of PD experiments are generally ineffective. Consider for example a common mode of PD where pedagogical theories supplemented with research findings are taught within in-service settings. From our perspective, there is no concretisation to objectivise the theoretical ideas advocated. Teachers are unlikely to translate these ideas easily into forms useable in the classroom. Another common PD mode involves the provision of detailed instructional package consisting of teacher guides, lesson plans, and lesson materials to support teacher implementation of a novel programme. This mode is usually linked to reforms supported by policy makers for implementation at scale. While it may appear that the instructional package is a form of concretisation, it does not cohere with our characterisation of concretisation as necessarily a negotiation of *teacher goals*. In the final analysis, many ambitious reform efforts fail because teacher goals are not given its rightful place in the scaling process.

But our contribution in this paper goes beyond the mere introduction of the construct of concretisation. We also purported design principles for a concretisation that would strengthen its potential to fulfil the roles intended. These principles can inform re-emergent fields of research and development work in textbook analyses and use of instructional materials. As an example, PD work that focuses on helping teachers learn how to use textbook materials—a strategy that is ostensibly 'concrete'—may still fall short in terms of implementation fidelity in the classroom (McNaught, Tarr, & Sears, 2010). Here, inquiry using the design principles may provide helpful analyses. Are the textbook materials designed in a way that help teachers integrate them into their existing instructional practices? Do the resources provide a visible form that is easy-to-use for teachers and helpful to students in viewing the underlying mathematical processes? We think that, in general, textbook designers focus heavily on comportment (that the resources 'carry' the reform intents) and accessibility (in terms of materials that are user-friendly to students); less attention is given to orienting the resources according to teachers' effective use – such as in the areas of integration and visibility.

Conclusion

In this "era of the teacher" (Sfard, 2005, p. 409), so much hinges on the teacher to render educational reforms 'alive' in the place where it really matters—the classroom. While the 'answer' to this challenge is often touted as teacher PD, there is still much to learn about how teacher learning experiences in PD settings can be translated into useable forms for effective classroom enactments. In this regard, we think that the conceptual framework that we proposed in this study (*Figure 1*) is a useful starting point to discuss the basic elements required to close the PD-Classroom gap. Furthermore, the construct concretisation can potentially specify strategies to inform designs of PD for impact in actual classroom practice.

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[1] Names of schools, teachers, and students that appear in this paper are pseudonyms.

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