Using the Computer to Diagnose Students' Difficulties with Fractions

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ABSTRACT

The objectives of the present study are to develop an automated computer system for diagnosis and remediation and to construct a conceptual model of remediation in fractions. The initial stage of the project began with the construction of an instrument to investigate the subjects' performances in addition of fractions. The items were based on a set of four objectives in addition of fractions. The test was administered to 3000 subjects who were classified as below average in fractions. The test was readministered a week later. The responses of the subjects were analysed and systematic errors were classified. An automated computer system for diagnosis and remediation in addition of fractions was developed. It consisted of three subsystems, viz. (a) diagnostic system of errors, (b) drill and practice for remediation, and (c) automated generation of text materials for remediation. To accompany the computer system for diagnosis and remediation, a conceptual model on remediation of addition of fractions was developed which was based on the hypothetical remedial activities.

Diagnosis and remediation in the teaching of mathematics have been seen by many teachers as essential for effective teaching. Okey (76) reported that pupils' achievements tended to go up when teachers gave diagnostic tests frequently. Although the use of diagnosis and remediation in teaching seems to be encouraging, the amount of time required for implementing the test and analysing the data to find out the actual causes of pupils' errors is tremendous. Unless the amount of time required can be reduced, teachers will normally be reluctant to carry out this strategy to help their pupils. Another factor which cannot be ignored is to determine the accuracy of diagnosing pupils' errors in mathematics. It would obviously be futile to conduct remedial classes which are mainly based on erroneous diagnosis. In view of these problems in teaching mathematics, there is a need for mathematics educators to consider the use of available technological tools to help teachers reduce their burden in the diagnosis and analysis of data. The topic of addition of fractions is here used as an example.

Review of Literature

'Diagnosis and remediation' is not something new in the mathematics education curriculum. However, research using the computer (especially the microcomputer) to help in diagnosing and remediating pupils with mathematics difficulties is not numerous.

Basically, the concept and work done in the area of diagnosis are pursued in two directions. The first group of mathematics diagnosticians concentrated their work on categorising the types of errors according to some major classifications. Robert (1968) classified four error categories, viz. wrong operation, obvious computation error, defective algorithm and random response. The work of Engelhardt

(1977), Cox (1975) and Knifong (1980) were quite close to Robert's work on errors analysis. This area of research was found to have two limitations. First, researchers tended to emphasise written responses from pupils and there had been few attempts to analyse pupils' difficulties by talking to them. Second, emphasis was placed on difficulties related to a type of mathematical task rather than a whole range of difficulties which pupils experience.

Error Patterns in Computation

Ashlocks (76) has in its turn given rise to another group of mathematics diagnosticians. Methods for correcting pupils' errors in computations were suggested in his book. Brown and Burton (1978) constructed a diagnostic model of subtraction using a representation technique called 'procedural networks'. Using this diagnostic model, two computer-based systems, BUGGY and DEBUGGY, were developed to teach both students and student teachers about the strategies for diagnosing bugs. Later, Brown and Van Lehn (1981) introduced the Repair Theory in an attempt to explain how the bugs (systematic errors) were acquired by students and how they were held. Travis and Carry (1983) and Woerner (1980) did similar research work to identify students' errors in multiplication and addition of fractions respectively. Travis and Carry concluded in their study that the diagnosis-remediation combination was effective for remediating students' errors in multiplication. Woerner concluded that the use of computer for diagnosis was effective when probing for more information. Bright (1984) suggested that future computer-based diagnostic systems should incorporate CAI for remediation.

Objectives

In view of the previous research and suggestions discussed in the previous paragraphs, a research project was initiated to investigate further into addition of fractions. The main objectives of the research study were to:

(1) classify a near-exhaustive set of error patterns in addition of fractions;
(2) develop a computer system for:
   (a) analysing of pupils' erroneous algorithms in addition of fractions.
   (b) generating drill and practice questions in remediation.
   (c) generating text materials for remediation; and
(3) derive a conceptual model for remediation in addition of fractions.

Methods

Sample

The sample for this study consisted of about 3000 average and below average pupils from 30 schools in Singapore. They were selected from the Primary 5 and 6 of the Normal Stream and the Primary 6, 7 and 8 of the Extended Stream (pupils take 6 years and 8 years to complete the Primary Education in the Normal and Extended Streams respectively).

Instrument

A diagnostic test on the addition of fractions was constructed which was based on the predetermined objectives. The four objectives identified for the test were (denominator $\leq$ 12):

(1) Addition of simple fractions with like denominators.
(2) Addition of simple fractions with unlike denominators.
(3) Addition of mixed numbers with like denominators.
(4) Addition of mixed numbers with unlike denominators.

In each objective identified above, four parallel items were used to test the subjects' knowledge in the algorithmic skills. This was to ensure that the different types of errors were identified, viz. systematic errors and non-systematic errors due to misreading a question or guessing a solution.

Procedure

The above diagnostic test was administered to the 3000 subjects. The subjects were retested in the following week. This was to ensure that systematic errors were identified. According to

Brown and Van Lehn (1984), systematic errors were made if the same pupils made similar mistakes in both tests. In both tests, no time limit was imposed on the subjects. However, they were expected to complete the test within a reasonable time. They were told to hand in their answer papers as soon as they had finished their work. Pupils' responses to each item of the tests were marked. Incorrect responses were carefully analysed to determine the actual error pattern of each mistake. Subjects were also interviewed when their errors made were randomised or they would be asked to think aloud on working a similar problem. The results obtained in the second test were used to check whether the erroneous strategies used by the subjects were systematic.

Results

One of the main objectives of this study was to derive a set of near-exhaustive systematic errors from pupils' errors. The results of these systematic errors were obtained by counter-checking the similar errors made in both tests. The results showed that not all errors made by pupils in both occasions were systematic. In categorising the types of errors made by the pupils, systematic errors were separated from the non-systematic errors. The table below summarises the various error patterns identified in each objective tested in the topic of addition of fractions.

Computer System for Diagnosis and Remediation

Owing to the nature of the topic on fraction, it was not the intention of this study to construct a procedural network to show a general diagnostic model in fractions. However it was found that, on the average, about 8 error patterns were identified in each objective.

TABLE 1: PUPILS' ERROR PATTERNS IN ADDITION OF FRACTIONS

(a)

Objective (1) : Addition of simple fractions with like denominators ($d \leq 12$)	
General Solution : $\frac{N_1}{D} + \frac{N_2}{D} = \frac{N_1 + N_2}{D}$	
Error Patterns	
(1) Adding numerators and denominators correspondingly	$\frac{N_1}{D} + \frac{N_2}{D} = \frac{N_1 + N_2}{D + D}$
(2) Multiplying the numerator instead of adding them	$\frac{N_1}{D} + \frac{N_2}{D} = \frac{N_1N_2}{D}$
(3) Treating the operator '+' as 'x'	$\frac{N_1}{D} + \frac{N_2}{D} = \frac{N_1N_2}{DD}$
(4) Carrying out cross-multiplications	$\frac{N_1}{D} + \frac{N_2}{D} = \frac{N_2D}{N_1D}$ or $\frac{N_1}{D} + \frac{N_2}{D} = \frac{N_1D}{N_2D}$
(5) Adding the denominators instead of the numerators	$\frac{N_1}{D} + \frac{N_2}{D} = \frac{N_1}{D + D}$
(6) Adding the denominators instead of the numerators	$\frac{N_1}{D} + \frac{N_2}{D} = \frac{N_2}{D + D}$

(b)

Objective (2) : Addition of simple fractions with unlike denominators ($d \leq 12$)	
General Solution : $\frac{N_1}{D_1} + \frac{N_2}{D_2} = \frac{N_1D_2 + N_2D_1}{D_1D_2}$	
Error Patterns	
(1) Adding numerators and denominators correspondingly	$\frac{N_1}{D_1} + \frac{N_2}{D_2} = \frac{N_1 + N_2}{D_1 + D_2}$
(2) Treating the operator '+' as 'x'	$\frac{N_1}{D_1} + \frac{N_2}{D_2} \neq \frac{N_1N_2}{D_1D_2}$ or simplified form
(3) Carrying out an incomplete algorithm	$\frac{N_1}{D_1} + \frac{N_2}{D_2} \neq \frac{N_1D_2 + D_1}{D_1D_2}$ When $N_2 \neq 1$
(4) Carrying out an incomplete algorithm	$\frac{N_1}{D_1} + \frac{N_2}{D_2} = \frac{D_2 + D_1N_2}{D_1D_2}$ When $N_1 \neq 1$
(5) Adding the denominators instead of the numerators	$\frac{N_1}{D_1} + \frac{N_2}{D_2} = \frac{N_1}{D_1 + D_2}$
(6) Adding the denominators instead of the numerators	$\frac{N_1}{D_1} + \frac{N_2}{D_2} = \frac{N_2}{D_1 + D_2}$
(7) Treating the operator '+' as 'x'	$\frac{N_1}{D_1} + \frac{N_2}{D_2} = \frac{N_2N_2}{D_1D_2}$
(8) Taking the bigger denominators as the common denominator	$\frac{N_1}{D_1} + \frac{N_2}{D_2} = \frac{N_1 + N_2}{D_1}$ $D_1 > D_2$
(9) Taking the bigger denominator as the common denominator	$\frac{N_1}{D_1} + \frac{N_2}{D_2} = \frac{N_1 + N_2}{D_2}$ $D_2 > D_1$
(10) Multiplying the denominators as LCM but numerators remain unchanged	$\frac{N_1}{D_1} + \frac{N_2}{D_2} = \frac{N_1 + N_2}{D_1D_2}$ or $= \frac{N_1 + N_2}{D_2}$ if $D_2 = nD_1$ or $= \frac{N_1 + N_2}{D_1}$ if $D_1 = nD_2$
(11) Multiplying the denominators as LCM but the numerator of the result equals to the sum of numerators and denominators	$\frac{N_1}{D_1} + \frac{N_2}{D_2} = \frac{(D_2 + N_1) + (D_1 + N_2)}{D_1D_2}$

(c) Objective (3) : Addition of mixed numbers with unlike denominators ($d \leq 12$)

General Solution : $W_1\frac{N_1}{D} + W_2\frac{N_2}{D} = (W_1+W_2)\frac{N_1+N_2}{D}$

Error Patterns

(1) Adding the whole number parts, the numerators and the denominators of the fractional parts correspondingly	$W_1\frac{N_1}{D} + W_2\frac{N_2}{D} = (W_1+W_2)\frac{(N_1+N_2)}{(D+D)}$
(2) Adding the whole number parts only	$W_1\frac{N_1}{D} + W_2\frac{N_2}{D} = (W_1+W_2)$
(3) Adding the fractional parts only	$W_1\frac{N_1}{D} + W_2\frac{N_2}{D} = \frac{(N_1+N_2)}{D}$
(4) Treating the whole number as the ten of the numerator	$W_1\frac{N_1}{D} + W_2\frac{N_2}{D} =$ $\frac{(10W_1+N_1 + (10W_2+N_2)}{D}$
(5) Multiplying the whole number and the numerator	$W_1\frac{N_1}{D} + W_2\frac{N_2}{D} = \frac{W_1N_1 + W_2N_2}{D}$

(d) Objective (4) : Addition of mixed numbers with unlike denominators ($d \leq 12$)

General Solution : $W_1\frac{N_1}{D^1} + W_2\frac{N_2}{D_2} = (W_1 + W_2)\frac{(N_1D_2 + N_2D_1)}{D_1D_2}$

Error Patterns

(1) Adding the whole number, the numerators and the denominator correspondingly	$W_1\frac{N_1}{D_1} + W_2\frac{N_2}{D_2} = (W_1+W_2)\frac{(N_1+N_2)}{(D_1+D_2)}$
(2) Adding the whole number and multiplying the fraction parts	$W_1\frac{N_1}{D_1} + W_2\frac{N_2}{D_2} = (W_1+W_2)\frac{N_1N_2}{D_1D_2}$
(3) Multiplying the whole number, the numerator and the fractional parts correspondingly	$W_1\frac{N_1}{D_1} + W_2\frac{N_2}{D_2} = (W_1W_2)\frac{(N_1N_2)}{(D_1D_2)}$
(4) Taking the bigger denominator as the common denominator and adding the whole numbers and the numerators correspondingly	(a) $W_1\frac{N_1}{D_1} + W_2\frac{N_2}{D_2} =$ $(W_1+W_2)\frac{(N_1+N_2)}{D_1}$ if $D_1 > D_2$ or (b) $W_1\frac{N_1}{D_1} + W_2\frac{N_2}{D_2} =$ $(W_1+W_2)\frac{(N_1+N_2)}{D_2}$ if $D_2 > D_1$

(5) Adding the unchanged numerators	(a) $W_1\frac{N_1}{D_1} + W_2\frac{N_2}{D_2} = (W_1+W_2)\frac{(N_1+N_2)}{D_1D_2}$ if $D_1 \neq nD_2$ or $D_2 \neq nD_1$ (b) $= (W_1+W_2)\frac{(N_1+N_2)}{D_2}$ if $D_2 = nD_1$ (c) $= (W_1+W_2)\frac{(N_1+N_2)}{D_1}$ if $D_1 = nD_2$
(6) Adding without using the whole number parts	(a) $W_1\frac{N_1}{D_1} + W_2\frac{N_2}{D_2} = \frac{N_1D_2+N_2D_1}{D_1D_2}$ if $D_1 \neq nD_2$ or $D_2 \neq nD_1$ (b) $W_1\frac{N_1}{D_1} + W_2\frac{N_2}{D_2} = \frac{nN_1+N_2}{D_2}$ if $D_2 = nD_1$ (c) $W_1\frac{N_1}{D_1} + W_2\frac{N_2}{D_2} = \frac{N_1+nN_2}{D_1}$ if $D_1 = nD_2$
(7) Treating the whole number as the ten of the numerator	$W_1\frac{N_1}{D_1} + W_2\frac{N_2}{D_2} = \frac{(10W_1+N_1) + (10W_2+N_2)}{D_1D_2}$

It was envisaged that teachers would find it difficult to memorise all these error patterns. Besides, it was also time consuming to analyse an individual's error in performing operations in fractions and other topics. Hence an automated computer system was developed to reduce the burden of teachers who would, presumably, be reluctant to perform the above tasks without such a system.

The Automated Computer System consisted of three subsystems. They were:

(1) Diagnostic System of Errors in Addition of Fractions.
(2) Drill and Practice for Remediation in Fractions.
(3) Automated Generation of Text Materials in Fraction for Remediation.

The Diagnostic System of Errors in Fractions was a system that would generate random questions which were based on 4 predetermined objectives. It could also determine the subjects' erroneous strategies in performing the addition of fractions. The subject was expected, if desired, to work out the problem on a piece of paper. The answer would be keyed into the computer and it would logically analyse the subjects's work and the probable cause of error would be printed out. The following tables show an example of an examinee's performances printed out from this computer system.

Table 2: Analysis of a Pupil's Performance in Fraction

Objective (3): Addition of mixed numbers with like denominators

Item 1 :

(a) Time taken : 9 secs

(b) Question : $2\frac{3}{11} + 2\frac{7}{11} = 4\frac{10}{22}$

(c) Error Pattern : Add the whole numbers, the numerators and the denominators correspondingly.

The Drill and Practice for Remediation is a system that generates random questions for drill and practice. The system was used to provide questions for drill and practice after the subjects had undergone remedial lessons conducted by the remedial teachers. The following table shows an example of the printout which summarises the examinees' performances after the subject had completed his drill and practice session.

The Automated Generation of Text Materials in Fractions for Remediation was designed to generate additional materials for the subjects to practise at home. The answers were also provided for the subjects to check the accuracy of their work.

Model for Remediation in Fractions

An overview of the error patterns made by the subjects in this study showed that most of the errors made were rudimentary. Brown and Van Lehn's Repair Theory could be used to explain the occurrences of the bugs. The examinees tended to apply a simpler strategy to work out the algorithmic operation.

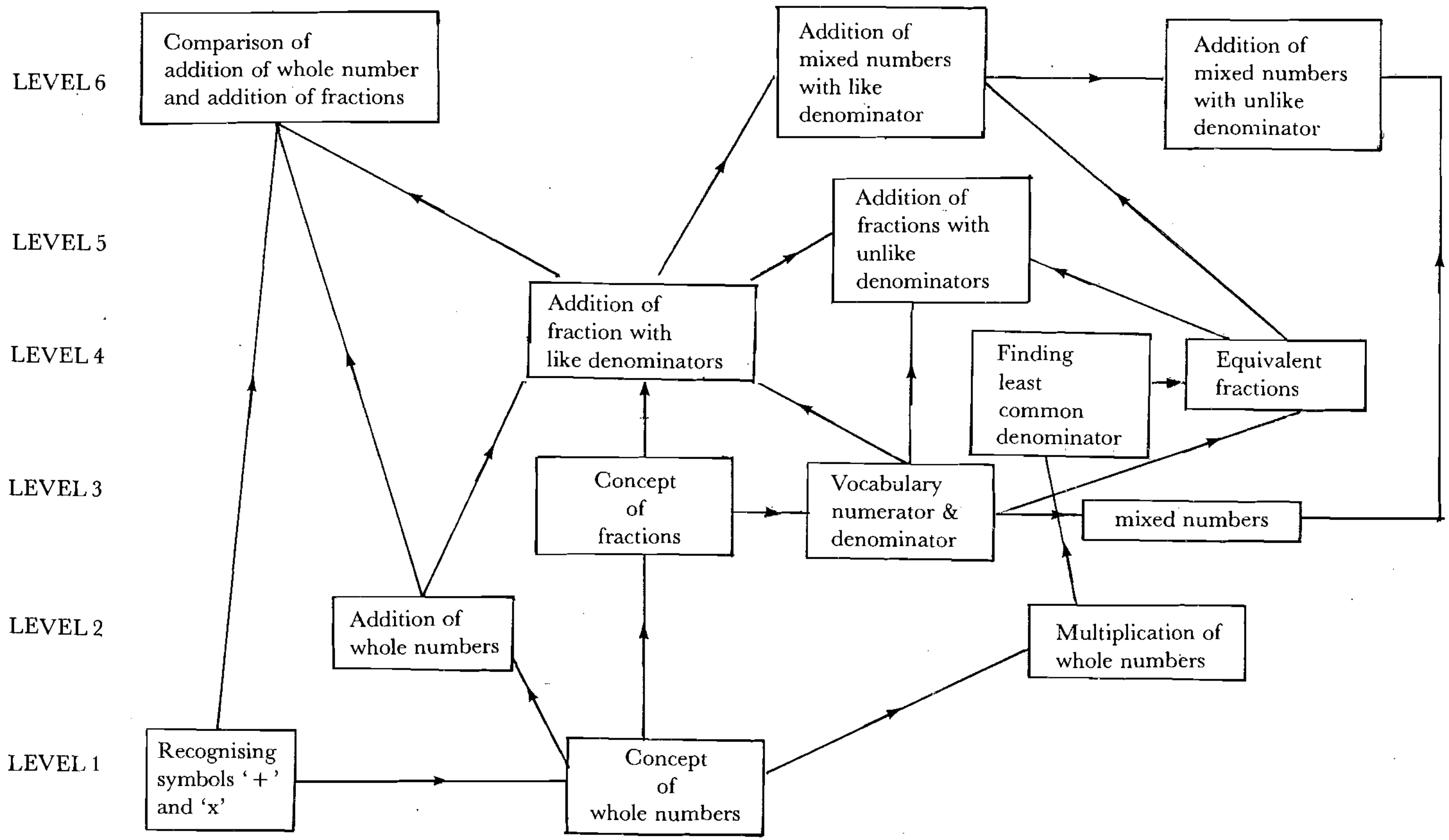
TABLE 3: SUMMARY OF RESULTS

Objective No.	Item No.	Result	Time Taken (sec)
1	1	Wrong	16
	2	Wrong	6
	3	Wrong	7
	4	Wrong	8
2	1	Wrong	10
	2	Right	7
	3	Wrong	6
	4	Wrong	5

TABLE 4: SUMMARY OF THE PUPIL'S PERFORMANCES IN FRACTIONS

Objective No.	No. of Question	No. Right	No. Wrong	% Right	Time Taken (sec)
1	10	5	5	50	12
2	10	6	4	60	23
3	10	3	7	30	13
4	10	2	8	20	43
Total	40	16	24	40	91
Average					22

FIGURE 1: CONCEPTUAL MODEL FOR REMEDIATION OF ADDITION OF FRACTIONS



An analysis of each error pattern was carried out and it was found that it had its own error identity. On the basis of its uniqueness, a set of hypothetical remedial activities for addition of fractions was suggested that would most likely alleviate the weaknesses of the subject. Using these remedial activities, a conceptual model for remediation of addition of fractions was constructed as shown in figure 1 on the previous page.

Each remedial activity is placed at one of the six levels identified. To help teachers identify the exact level of non-achievement, the computer system could be designed to print out the required level for remediation. Using this conceptual model for remediation of addition of fractions, teachers would be able to select a set of those remedial activities classified at and below the level identified by the computer system for carrying out remedial work. For example, the computer system could pick out the pupil's error and indicate that remedial work would be required to cater for Level 3 (concept of fraction). Hence the set of possible remediate activities would then be comprised of activities at Level 3 and below.

Conclusion

Two important outcomes emerge out of this study, viz. development of an automated computer system for diagnosis and remediation activities for addition of fractions. This computer system and the conceptual framework for remediation provide an alternative approach for individualising instruction in mathematics. It serves as a prototype system to cater for other areas of mathematics in diagnosis and remediation.

This system does not only provide teachers with an accurate diagnosis of errors but it also helps to reduce the time required to analyse examinees' errors. With the remedial information printed out, the investigator may conduct remedial activities immediately without wasting much time in looking for remediation materials. In the process of using the system for diagnosis, the investigator may also be able to collect further information on error patterns as the set of error patterns identified earlier may not be exhaustive. This provides additional information for research.

Two assumptions have been made in this study. The hypothetical remedial activities are assumed to be effective and exhaustive. Further research should concentrate on verifying the remediation model and the accuracy of the diagnostic system.

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