Title	Are patterns important? An investigation of the relationships between proficiencies in patterns, computation, executive functioning, and algebraic word problems
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Source	*Journal of Educational Psychology,103*(2), May 2011, 269-281

This article is published in Lee, K., Ng, S. F., Bull, R., Pe, M. L., & Ho, R. H. M. (2011). Are patterns important? An investigation of the relationships between proficiencies in patterns, computation, executive functioning, and algebraic word problems. *Journal of Educational Psychology, 103*(2), 269-281, DOI: 10.1037/a0023068

Header: Patterns

This is an uncorrected version of the article. The published version can be found in the Journal of Educational Psychology (2011)
doi: 10.1037/a0023068

Are patterns important? An investigation of the relationships between proficiencies in patterns, computation, executive functioning, and algebraic word problems.

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Version date: 12 January 2011

Abstract

Although mathematical pattern tasks are often found in elementary school curricular and are deemed a building block for algebra, a recent report (US National Mathematics Advisory Panel, 2008) suggests there needs to be a re-balance in the resources devoted to its teaching and assessment. We examined whether children's developing proficiency in solving algebraic word problems is related to their proficiencies in patterns, computational, and working-memory tasks. Children (*N* =151, 10 year olds) were tested twice, one year apart, and were administered tests of updating capacities (two complex span and a running span task), computation (from the Wechsler Individual Achievement Test), patterns (function machine, number patterns), and algebraic word problems. Proficiencies on the patterns and computational tasks predicted algebraic proficiency. Proficiencies on the computational and patterns tasks are in turn predicted by updating capacity. These findings suggest that algebraic reasoning may be difficult if the child has poor updating capacity and either poor facility with computation or difficulty in recognizing and generalizing rules about patterns.

Keywords: Working memory, cognitive development, academic achievement, mathematics

Are patterns important? An investigation of the relationships between proficiencies in patterns, computation, executive functioning, and algebraic word problems.

One of the recommendations of the US Presidential Mathematics Advisory Panel (2008) was a relative reduction in the amount of attention devoted to *patterns* in curriculum and assessment. Patterns tasks vary from those involving simple repeating geometric shapes, e.g., □, ◊, ○, □, ◊, ○, □, ?, ○, to those involving basic numerical sequences, e.g., 2, 4, ?, 8,10, 12, 14, or more sophisticated numerical relationships, e.g., 0, 1, 1, 2, 3, 5, 8, ?, 21, 34. Such patterns have in common a fixed functional relationship between each succeeding symbol or number. The Panel cited international comparison, stating that with the exception of Singapore, other "A+" countries had seldom given emphasis to the teaching of patterns in the K-6 curriculum. Although the Panel only called for a relative reduction in weighting, it is unclear to what extent a curriculum focusing on developing algebraic thinking skills should focus on patterns.

Skills learned in solving patterns tasks are deemed important in preparing children for algebraic problem solving (Mason, 1996; Orton & Orton, 1999). However, a search of the individual differences literature revealed no empirical investigation of the relationship between proficiencies in patterns and algebraic problems. In our previous studies, we found individual differences in children's algebraic proficiency to be explained by executive functioning; in particular, children's capacity to update or to process and remember information simultaneously (Lee, Ng, & Ng, 2009; Lee, Ng, Ng, & Lim, 2004). In this study, we examined the extent to which proficiencies on patterns tasks explained individual differences in performance on algebraic word problems at Primary 4 and 5 (9 and 10 year olds). We also examined the extent to which proficiencies on patterns and algebraic tasks at Primary 5 are predicted by children's prior proficiencies in updating and other domain specific skills.

Patterns, Arithmetic Computation, & Algebra

Orton and Orton (1999) described number patterns as a popular tool for developing children's ability to express generality, which is argued to be one of four routes to algebra (Mason, 1996). They found that children, faced with number sequences, tended to focus on the numeric distances between all or a subset of numbers in the sequence. They then used this information to generate additional items in the sequence. From this perspective, success with discerning numeric distances and generating additional numbers based on a sequence are likely reliant on a sound knowledge of arithmetic computation.

In pattern tasks, problem solvers are presented with a sequence of numerical stimuli and are asked to deduce the missing items. Take for example a number sequence involving 42668, 43668, ? , 45668, 46668. Some problem solvers will start by focusing on the first two numbers and once they have a sense of the structure underpinning the pattern, determine what remains the same and what changes (Mason, 1996). The next step involves constructing a rule that enables them to extend the number sequence, and to end with the final number 46668. In our example, generation of the next number in the sequence requires applying the rule, +1000 to the preceding number. Some will test the rule by undoing the process: If +1000 allows the sequence to be generated, undoing the rule, -1000, should generate the preceding numbers. Solution of such number pattern tasks will likely depend on sound facility with arithmetic operations and number facts, as well as the pellucidity of the underlying mathematical pattern to the problem solvers.

Patterns to algebraic word problems. An issue that remains unclear is how solving patterns and algebraic word problems are related. Are the same processes involved in the completion of a number sequence, generating a rule governing a function, and computing the solution to an algebraic word problem? The importance of patterns for algebra is not readily apparent from a comparison of their task requirements. In number pattern tasks, problem solvers are tasked to identify the relationship between consecutive items. In algebra word problems, the relationships between the variables (i.e. protagonists) as well as the final output are given. The challenge is to work out the value of each variable. In what way then can proficiency or exposure to patterns facilitate the acquisition of algebra problem solving skills?

With number pattern tasks, a sequence of numbers with one of the number missing is given (e.g., $T_{1,}$ $T_{2,}$ T_3, ..., T_{n-2}, T_{n-1}, T_n). The first input term, T_1 is succeeded by *n* other terms used to specify the sequence. Regardless of the position of the missing term, the objective of the problem solver is to ascertain the relationship underpinning this sequence of numbers and work out the missing term. For example, if the relationship between pairs of consecutive terms is a constant difference *d*, then $T_2 – T_1 = T_3 – T_2 = T_n – T_{n-1} = d$. If T_{n-1} is the missing term then $T_{n-1} – T_{n-2} = d$, and $T_{n-1} = T_{n-2} + d$.

Although algebraic word problems tend to be more challenging than number pattern tasks, the two types of questions can be structurally similar. This is particularly the case for simple algebraic questions in which protagonists differ by a constant difference. Take, for example, the following question.

> "Joshua and Mary have $86,336; Mary has $1000 more than Joshua. How much does Mary own?"

If the amount of money held by Joshua (T_J) and Mary (T_M) are viewed as being equivalent to the consecutive terms in a number sequence task, we can readily see its relationship with pattern tasks. The values of both terms, T_M and T_J, are unknown, but the relationship between them is specified in the question and can be represented as $T_M - T_J = d$ (in this case, $d = \$1000$).

What the two tasks have in common is that they can both be characterised as instantiation of $T_n = T_{n-1} + d$, where n refers to the number of terms in a number sequence or one of the protagonists in an algebra question. In number sequences, *d* refers to the absolute or functional difference between two consecutive terms. In algebra questions, *d* is given and refers to the relationship between two unknown inputs. Early exposure to pattern tasks may sensitise children to the relationships between the known and unknown in an algebra question and make the problem solving processes easier to understand. Take, for example, the use of substitution to solve for the unknown. When problem solvers are told $T_J + T_M = S$ (where S is a sum; $86,336 in our example) and T_M differs from T_J by *d*, they can solve the question by substituting T_M with $T_J + d$. Generating $T_J + (T_J + d) = S$ is a significant step in the problem solving process and requires an understanding of the relationship between part and whole, as well as the importance of symmetric and transitive equivalence: all important concepts in algebra. Experience working with number patterns may give children a head start as they provide exposure to symbolic manipulation and the notion of equivalence, which are also required for solving algebraic word problems.

Algebraic word problems to patterns? Although the dominant discourse in the mathematics education literature focuses on how exposure to patterns facilitates algebraic thinking, is it also possible that facility with algebraic thinking assists in children's ability to grapple with patterns? From an epistemological point of view, this seems unlikely. In curricula with which we are familiar, symbolic algebra is not introduced until the high school years. By which time, children have already had repeated exposure to pattern tasks. Algebraic thinking tasks are introduced earlier in some countries, but they are usually predated by exposure to pattern tasks. Yet, when development is considered from a point in time at which children had been exposed to both patterns and algebraic thinking, it seems possible that knowledge of algebra will reinforce that of patterns. As argued above, solving pattern and algebra tasks may involve some of the same processes; pre-existing knowledge of one may assist in the other. In this study, we used longitudinal data, collected one year apart, to examine whether individual differences in a pattern task are explained by prior performances in an algebraic task, and vice versa.

The Role of Executive Functioning

Executive functions refer to cognitive processes that control, direct, or coordinate other cognitive processes. One such function that is often studied is updating. Updating refers to refreshing the content of information in active memory with newer or more relevant information. It also involves maintaining information in a readily available state. It has been found to be closely related to working memory, as measured by complex span tasks (Miyake et al., 2000). In the literature, the terms updating and working memory are sometimes used interchangeably. On a theoretical level, finer distinctions can be made depending on whether one understands working memory from a structural (Baddeley & Hitch, 1974) or functional (e.g., Cowan, 1999; Engle, 2002) perspective. Working memory typically refers to a more expansive set of processes or structures that allow for information to be maintained in a readily accessible state while being processed.

In an earlier study, Holzman, Pellegrino, and Glaser (1983) found college-aged adults' and 10-year-olds' performances on pattern tasks to be affected by the working memory demands of the tasks. They also found performance to be affected by the participants' skills in arithmetic computation and in dealing with hierarchical relations. Looking further afield, there is a wealth of studies on the relationship between inductive reasoning and working memory. Kail (2007), for example, found improvement in inductive reasoning to be driven by developmental increases in working memory amongst 8- and 13-year-olds.

In the mathematical cognition literature, a number of studies have shown that the algebraic proficiency of primary school children is related to working memory or updating capacity. In an earlier study (Lee et al., 2004), we found working memory predicted individual differences in proficiency in solving algebraic word problems. Furthermore, the relationship was found to be independent of children's language proficiency and non-verbal intelligence. There are several explanations for the inter-relation between working memory capacity and algebraic proficiency. First, children have to access what they know about algorithms for solving such problems and imposing this information on the question at hand. Acting as a buffer between short and long term memory is a common feature of many working memory models (for a review, see Conway, Jarrold, Kane, Miyake, & Towse, 2007; Miyake & Shah, 1999). Second, even with relatively straightforward questions, children have to think about how quantitative relationships specified in a word problem can be translated into a mathematical statement. Take for example, if Mark has five balls more than Jane, should it be translated into M + 5 = J, or J + 5 = M? Coming to a decision will presumably involve holding both representations in mind and considering their suitability. Remembering and processing information at the same time is a key function of working memory. Indeed, in a study examining the working memory requirements of different components of algebraic problem solving, Lee, Ng, and Ng (2009) found translating quantitative relationships from words to mathematical operations to be particularly resource intensive.

Although the role of updating or working memory is relatively clear if problem solvers were to pause and consider the structural relationship between protagonists before proceeding to computation, children do not always engage in such deliberation. Many children are accustomed to picking out keywords (e.g., more than or less than) and automatically translating them into mathematical operators (addition and subtraction respectively). Given the high level of homogeneity found in many mathematics textbooks (Mayer, 1981), this strategy often works. It does not work when questions are stated in ways contrary to convention, e.g., “Mark has 9 balls. He has 3 balls fewer than Jane. How many balls does Jane have?” With such questions, ability to inhibit the tendency to map “more than” or “less than” in the accustomed manner seems vital to success. Inhibitory efficiency has been found in some studies to be closely related to updating capacity (Miyake et al., 2000).

Updating or working memory is also involved in arithmetic computation. In a conventional school algebra problem, arithmetic computation is always the final step in producing a solution. Evidence from both correlational and experimental studies have shown that access to working memory resources is needed for performing computational tasks. In Andersson (2008), for example, tasks designed to index the central executive component of working memory predicted performance on a written arithmetic task independently of reading, age, and intelligence. In Lee, Ng, and Ng (2009), performance on updating measures predicted performance in a computational task. Using an experimental approach, Fürst and Hitch (2000) showed that the carrying or regrouping component in multi-digit mental arithmetic task imposed heavy demands on executive resources. Having to perform the arithmetic task together with a secondary task known to draw heavily on executive resources resulted in both slower performance and more errors than performing the same arithmetic task with articulatory suppression or no secondary task (also see Imbo & Vandierendonck, 2007; Imbo, Vandierendonck, & De Rammelaere, 2007).

Research Questions and Hypotheses

Given the curricular implication, it is important to ascertain the extent to which proficiency in algebraic word problems is reliant on proficiency and prior exposure to patterns. We examined this issue using structural equation modelling with data collected from children in the third term of Primary 4 (~ 9.5 years old). In Singapore, children are introduced to algebraic word problems at the beginning of Primary 4. Although children are not taught to construct and manipulate algebra equations at this age, they are taught to use pictorial representations to model word problems. It is the construction of these pictorial models that is believed to require algebraic thinking (Ng & Lee, 2009).

Although there is a dearth of prior studies on this issue, our reading of the mathematical education literature led us to expect a strong positive relationship between proficiency in patterns and algebraic tasks (see Figure 1, this relationship is labelled R1). Findings from Orton to Orton (1999) showed that children relied on arithmetic computation to solve pattern tasks, we examined the extent to which this reliance explained individual differences in performance on the pattern tasks (labelled R2 in Figure 1). We also estimated the relationships between computational, pattern, and algebraic proficiencies simultaneously to examine the possibility that proficiency in arithmetic underlies the relation between pattern and algebra (labelled R3).

Insert Figure 1 about here

There are both correlational and causal demonstrations of relationships between updating/working memory and different aspects of algebraic problem solving (Lee et al., 2009; Lee & Ng, 2009). In this study, we focused on updating. To understand better the contributions of patterns and computation to algebra vis-a-viz domain-general capacities, we examined whether relationships between patterns, computation, and algebra are mediated by relationships with updating (labelled R4 in Figure 1). Being domain-specific, performances on both the patterns and computational tasks were expected to be correlated strongly with accuracy on the algebraic task. We also expected children with larger updating capacity to perform better on both the patterns and computational tasks. We were particularly interested in whether updating, having controlled for the contributions of patterns and computation, would still have a direct effect on algebraic performance.

To test for replicability, we tested the structural models with data from a second cohort of Primary 4 children. We also tested the original sample one year later, at Primary 5. In the Singapore mathematics curriculum, curricular material are introduced, then revisited and built upon in subsequent years. The longitudinal data allowed us to examine the extent to which prior proficiencies in patterns and computation are important for subsequent performance on algebraic questions (denoted by dashed directional arrows in Figure 1). It also allowed us to test the directionality issue that was raised earlier; specifically, whether it is exposure to patterns that affects algebraic performance, or vice versa (denoted by dash-and-dot lines in Figure 1). The data also allowed us to examine the extent to which findings identified at Primary 4 are replicated at Primary 5. In a previous study, computational fluency was found to explain no additional variance in an algebraic task when proficiencies in understanding the algebraic question and translating the question into a pictorial representation were statistically controlled (Lee et al., 2009). It was argued that by Primary 5, children have such familiarity with arithmetic that it is no longer an obstacle to good performance. As arithmetic competency takes time to build, it is of interest whether there is a decrease in reliance on such competency from Primary 4 to 5.

Method

Participants

Children were recruited as part of a larger cross-panel longitudinal study examining the relationship between the development of executive functioning and mathematics proficiencies. Children were recruited via parental consent letters sent to five participating government funded schools. All schools were located in middle to lower middle class areas in western Singapore. With all non-language specific lessons conducted solely in English, by Primary 4, pupils could be considered functionally bilingual.

We focused on data from 151 children who were first recruited into the study when they were in Primary 4 (M_{age} = 120.58 months, *SD* = 3.59, 74 boys). Data from this first wave and a second wave, collected one year later, are reported. Due to absences from school and logistical constraints that prevented us from testing the children, 1.47% of the data were missing (spread over 12 children). When we retested the children one year later, 9 children had withdrawn from the study. Altogether, 8.15% of the data from the second wave were missing (spread over 35 children). To avoid a reduction in power, we used the full information maximum likelihood (FIML) approach, as implemented in AMOS 18 (Arbuckle, 2010), to calculate parameter estimates for the model. The benefit of this method is that statistical models are estimated using an iterative approach using all available information. Furthermore, unlike traditional approaches for dealing with missing data (e.g., listwise or pairwise deletion), missing-completely-at-random is not required for the method to produce unbiased estimates (Enders, 2006).

Materials and Procedure

Children were administered a large battery of executive functioning, reading comprehension, intelligence, motivation, and mathematics tasks. The tasks were divided into 5 sets and were administered over several sessions. Each set took approximately one hour. Separation between sessions varied depending on school schedule and children's availability. In most cases, sessions were conducted on consecutive days. In exceptional cases where we had limited access to the children, we ran two sessions per day, administering only a subset of the tasks. In this paper, we focused on children's performances on the updating/working memory, patterns, computational and algebraic word problem tasks.

The same set of updating and working memory tasks was used in both waves. We used the Listening Recall and Mr X tasks from the Automated Working Memory Assessment (Alloway, 2007a) and a pictorial updating task. Computational proficiency was measured by the Numerical Operations task from the Wechsler Individual Achievement Test, Version 2 (Wechsler, 1992). Algebraic word problems and patterns tasks, based on the Primary 4 and 5 curricula, were administered at Wave 1 and 2 respectively.

Updating tasks. Similar to factor analysis, structural equation modelling uses similarities between observed variables to generate latent factors. To increase the likelihood that what is captured is the similarity in the underlying construct being measured, rather than other surface characteristics, we selected a variety of working memory and updating tasks to index updating capacity. The two working memory tasks are often referred to as complex span tasks and were chosen because Miyake et al. (2000) found updating to be a key process that underlies performance in these tasks. In the Listening Recall task (Alloway, 2007a) children listened to a series of sentences and were asked to identify whether each sentence was true or false. At the end of each trial of sentences, the children were asked to recall the last word of each of the sentences, in the correct order. The test progressed from trials containing one sentence to trials containing six sentences. Each span or block of trials contained six trials. The measure used for working memory in this task is the total number of points received from recalling the final word in each of the sentences (range = 0 to 36, test-retest reliability = .81, Alloway, 2007b).

In the Mr. X task (Alloway, 2007a), the children were shown two Mr. X figures, each holding a ball at one of eight cardinal positions. They had to decide whether the figures were holding the ball on the same hand. At the end of each trial, the children had to point to the position at which each ball was held, in the correct order. The task progressed from a block containing one set of Mr. X figures to a block containing seven sets of figures. Each block contained six trials. The total number of positions recalled served as the dependent measure (range = 0 to 42, test-retest reliability = .77, Alloway, 2007b).

In the Pictorial Updating task, children were shown a series of animal pictures, one at a time. To ensure that updating was being used in the task, the children did not know how many items were going to be presented, and were asked to recall a specified number of animals from the end of each trial. The number of animals presented was varied randomly across trials (Min = 3, Max = 11). The task began with the children recalling the last two animals. This increased to the last four. Each block contained two practice sets and twelve experimental trials. The children received a point for every animal recalled correctly. The order of recall was not taken into account (range = 0 to 108, test-retest reliability $_{\text{one year}}$ = .58).

Computational Proficiency. In the Numerical operations task (Wechsler, 1992), children were asked to solve written computational problems. We followed the published standardised administration procedure. A point was given for every correct response. To facilitate comparison across age groups, in addition to their accuracy on items designed for Primary 5 children, we awarded to them scores from all questions meant for Primary 4 children (range = 0 to 54, P4: Kuder-Richardson Formula 20[1] (*KR*20) = .71; P5: *KR*20 = .83).

Patterns task. The Number Series task consisted of 24 items. The task was divided into three subparts: six items were drawn from the more difficult items in the previous year's

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[1] The Kuder-Richardson Formula 20 (Kuder & Richardson, 1937) is analogous to the more commonly used Cronbach α, but has the advantage that it can be used with dichotomous data. Like the Cronbach α, values range from 0 to 1 with larger values indicating a higher degree of homogeneity.

curriculum, 12 items were representative of the core content in the present year's curriculum, and the remainder were drawn from easier items in the subsequent year's curriculum. The number patterns were sequences of numbers, whole or rational, constructed using a particular mathematical rule. Children were expected to study the pattern in the sequence and find the missing number. Each correct response was given one point. To constrain the children's responses, the missing item was never placed at the end of a sequence. For the present study, we used only the 12 items representative of the year's curriculum (P4: range = 0 to 12, *KR20* = .90; P5: range = 0 to 12, *KR20* = .75).

In the second pattern task, Function Machine, children were asked to identify missing numbers and the rules that governed the relationships among pairs of input and output numbers. The function machine total score was comprised of three subscores based on children's abilities to generate: a) the missing input variables (P4: range = 0 to 30, *KR20* = .97; P5: range = 0 to 30, *KR20* = .92), b) the missing output variables (P4: range = 0 to 30, *KR20* = .96; P5: range = 0 to 30, *KR20* = .93), and c) identification of the rules relating the input to the output units (P4: range = 0 to 30, *KR20* = .97; P5: range = 0 to 30, *KR20* = .94).

Algebraic task. Items for the instruments used in the two waves were modified from Lee et al. (2009; 2004) and each contained 10 algebraic, start-unknown questions. To provide comprehensive coverage of the types of questions included in the curriculum, a range of questions utilising different quantitative concepts were used. For Primary 4, we used four questions involving the relational concepts of "more than" or "less than". In the next four questions, multiplicative relationships of the type "n times as many", where n was a whole number, were used to express the relationship between the two protagonists. The remaining two questions involved knowledge of and operations on fractions. We used the same structure to construct questions for Primary 5. The complexity of these questions was increased by increasing the number of protagonists from two to three. For questions involving multiplicative relationships, a combination of whole number and fraction multiplicative relationships were used. Responses were coded as either right or wrong (P4: range = 0 to 10, *KR20* = .94; P5: range = 0 to 10, *KR20* = .86).

Results

Measures were screened for missing values, outliers, and normality of distribution. Scores that were more than three *SD* beyond their respective means were replaced by values at three *SD*. The distributions of all variables were approximately normal.

At Primary 4, the bivariate correlations between the computational, pattern, and algebraic measures ranged from moderate to large. Correlations between the WIAT Numerical Operations and the two pattern tasks ranged from .42 to .60. Computational proficiency accounted for around a quarter of variance in performances on the pattern tasks. Correlations between the patterns and algebraic measures ranged from .50 to .67, with patterns proficiency accounting for roughly a third of the variance in performances on the algebraic questions.

Between the three updating measures, correlations were moderate ($.41 > r > .30$). Amongst the three measures, performance on the Pictorial Updating task was more strongly correlated with the various mathematical measures ($.46 > r > .26$) than were performances on Mr. X ($.29 > r > .12$) and Listening Recall ($.32 > r > .19$). At Primary 5, the three updating measures exhibited correlations that ranged from .26 to .35. Similar to data collected at Primary 4, correlations between the various mathematical measures and the Pictorial Updating tasks ($.52 > r > .37$) were higher than their correlations with the other working memory measures ($.42 > r > .15$). Full means and correlation ratios can be found in Table 1.

Insert Table 1 about here

We conducted several sets of structural equation modelling using AMOS 18 (Arbuckle, 2010). The first two examined the relationships between the various mathematical measures and between the updating and mathematical measures. Data from the Function Machine and

Number Series tasks served as indicators for Patterns. Pictorial Updating, Mr. X, and Listening Recall served as indicators for an Updating latent variable. To avoid ambiguity associated with the use of a latent factor generated from a single indicator, we divided questions from the WIAT Numerical Operations task into three groups (Questions 1, 4, 7 ... into Group 1; Questions 2, 5, 8 ... into Group 2; Questions 3, 6, 9 ... into Group 3). This measured the children's Computational Proficiency. For similar reasons, the algebraic questions were subdivided with each group containing questions of varying difficulties to produce three groups with similar group means. These served as indicators for the Algebra latent factor.

Performances in Wave 1

We tested four models on the Primary 4 data. In Model 1, we tested whether Updating, Computational Proficiency, and Patterns had direct effects on algebraic performance. We also tested whether Updating has a direct effect on Computational Proficiency and Patterns, and whether Computational Proficiency has a direct effect on Patterns (see Figure 2). Models 2 to 4 are tests of the assumptions that we built into Model 1.

Insert Figure 2 about here

The analysis showed that Model 1 provided a good fit to the data (see Table 2 for fit indices). All regression weights between indicators and their respective latent variables were significant. Updating explained a significant amount of variance in both Computational Proficiency (see Figure 2 for standardised parameter estimates and Table 3 for unstandardized estimates) and Patterns. Patterns explained a significant amount of variance in Algebra, as did Computational Proficiency. Children with better computational proficiency did better on the patterns task. Notably, Updating was not significantly related to Algebra. The overall model accounted for 63.4% of variance in Algebra.

Insert Tables 2 & 3 about here

In Model 2, we tested explicitly the relationship between Patterns and Algebra by constraining the path between the two variables to zero. In effect, this model is a test of the null hypothesis regarding the relationship between Patterns and Algebra. The model resulted in poorer model fit than the unconstrained model, thus providing further evidence that proficiency on the algebraic questions is dependent on performance in the pattern tasks (see Table 2 for model comparison statistics and Table 3 for changes in parameter estimates across models).

In Model 3, we examined whether a model in which Updating has no direct effect on Algebra provided a better fit to the data: the direct path from Updating to Algebra was constrained to zero. The model fit was similar to Model 1. Both models showed that Updating had no direct effect on Algebra.

In Model 4, we tested the assumption that performance on the patterns task is dependent on computational proficiency. We added to Model 3 an additional constraint that specified a null relationship between Computational Proficiency and Patterns. Its model fit was significantly poorer than that of Model 3 (see Table 2). These findings showed that Model 3 provided the best and the most parsimonious fit to the data. An examination of the total effects associated with the model showed that Computational Proficiency had the greatest effect on Algebra: an increase of 1 standard deviation in Computational Proficiency was associated with an increase of .73 standard deviation in Algebra. Although Updating had only indirect effect on the criterion, via Computational Proficiency and Patterns, its total effect (.51) was greater than that of Patterns (.28).

Replication. An adequate fit between model and sample data does not preclude the possibility of an equally good fit with another model. To test further the adequacy of Model 3, we fitted the same model to another cohort of 150 Primary 4 children (M_{age} = 120.33 months, *SD* = 3.63, 78 boys), who entered the study at Primary 2. These children were administered the same battery of tests two years after the original Primary 4 sample. Apart from belonging to a different cohort, these children differed from the original sample in that they had been administered the same updating measures on two previous occasions (when they were in Primary 2 and 3). Because all the mathematical items contained grade appropriate items, the majority of items used at each grade were novel. Similar to the original sample, we were unable to obtain data from some children due to absences from school and other logistical constraints. Overall, 1.48% of the data were missing (spread over 6 children). We again used the FIML approach, as implemented in AMOS 18 (Arbuckle, 2010), to deal with the missing data.

We examined whether Model 3, as fitted to the original sample, also provided a good fit to the data from this sample. We did so by fitting the same model to both samples and applying increasingly restrictive tests of invariance. In the first model (identical to Model 3), all parameters across the two samples were left unconstrained. In other words, we computed the best fitting estimates for each sample. This served as a baseline against which all other models were compared. In the second model, factor loadings of the various manifest measures on their respective latent factors were held constant across the two samples. This provided information on whether the latent factors (e.g., Updating, Patterns) were constituted in the same way across the two samples. In the third model, we held constant across the two samples, factor loadings as well as structural weights between the various latent factors. If this model does not produce a poorer fitting model than the baseline, relationships of dependence between the latent factors can be said to be similar across the two samples. Equality of covariance between the latent variables was added in the fourth model. This model imposed additional constraints on the correlational or bidirectional relationships. This was followed by equality constraints on the disturbances and measurement errors in the fifth and sixth models, respectively. These final constraints tested the degree to which the magnitude of error specific to each predicted latent factor and specific to each instrument replicated across the two samples.

Both changes in χ^2 and CFI (Cheung & Rensvold, 2002, or the practical difference approach, Byrne, 2010) showed invariance in parameters across the two samples down to the second most restrictive (fifth) model (see Table 2). This provided substantive evidence for the replicability of both the relationships of dependence and of the bidirectional relationships. Because invariance of test reliability is not a concern in this study, violation of equality at the measurement errors level (the sixth model) is deemed to be of little consequence (Byrne, 2010). These findings provide strong support for the applicability of Model 3.

Equivalent Models. A weakness of structural equation modelling is that for any given model, there may be equivalent models that cannot be distinguished by statistical means. These models provide the same estimated correlations and covariance, but with different configurations amongst the specified latent variables (Kline, 2005). In our model, for example, although a wealth of previous studies have established the dependency of Computational Proficiency on Updating, the relationship between Updating and Computational Proficiency can be reversed without affecting either model fit or the regression weights between the various latent factors. Reversing the hypothesised relationships between Patterns and Algebra, or Patterns and Computational Proficiency, resulted in identical model fit, but different regression weights amongst the latent factors. To examine further the extent to which algebraic performance can be explained by proficiency on the patterns and computational tasks, we turned to the longitudinal data to provide a temporal dimension to help us disambiguate these relationships.

Performances One Year Later

We first ran the same structural models as those used in Wave 1 to examine whether they are also appropriate for data collected one year later. Model 1 provided a good fit to the data (see the Primary 5 panel in Table 2 for fit indices). With the exception of the path from Updating to Algebra, all other structural paths were significant ($.70 \geq \beta \geq .40$). The model explained 68.4% of variance in algebraic proficiency. In Model 2, the path from Patterns to Algebra was constrained to zero. This produced a poorer fit and some irregularities in the structural parameters.

Constraining the path from Updating to Algebraic to zero in Model 3 had a negligible effect on model fit. Similar to findings from Wave 1, Model 4 had a poorer fit than Model 3. These findings showed that Model 3 also provided the best fit to the Wave 2 data.

Relationships across the Two Waves

To examine the extent to which proficiencies in Primary 4 affected performances one year later, we fitted all within-wave and cross-lag structural paths (e.g., from Updating Wave 1 to Computational Proficiency Wave 2) with the exception of those from Updating to Algebra, which were shown to be non-significant in the previous analyses[2]. The model showed a good fit to the data, $\chi^2(267) = 384.33$, $p < .01$, CFI = .96, RMSEA = .05 (PCLOSE = .28), and AIC = 604.33. Because a number of structural paths in this longitudinal model were non-significant; they were trimmed from the model. The final model (Model 5, see Figure 3) provided a good fit to the data (see the Primary 4 to 5 panel in Table 2).

Insert Figure 3 about here

All paths were significant and the model explained 84.8% of variance in the Wave 2 algebraic word problem task. With the exception of Patterns, all Wave 1 latent factors strongly predicted their Wave 2 counterparts. Notably, some of the strongest relationships were found

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[2] We initially fitted only structural paths from each Wave 1 latent factor to its corresponding Wave 2 factor. This resulted in a negative variance estimate for the disturbance term of the Wave 2 Updating factor. Inspection of the correlations suggests that this resulted from higher correlation within tasks across the two waves than amongst tasks within each wave. Because participants were given the same updating tasks across the two waves, it is not unreasonable for both updating and task specific variances to be correlated across the two waves. We accounted for this by fitting autocorrelated paths for the error terms.

between Computational Proficiency at Wave 1, its counterpart at Wave 2 and Patterns at Wave 2. These findings point to the importance of early competency in arithmetic computation. Though the relationships between Pattern and Algebra were significant at both time-points, Pattern at Wave 1 did not have a direct effect on Algebra at Wave 2. It only had an indirect effect via Wave 1 Algebra.

Equivalent models. To test the viability of paths suggested by the equivalent models, we added paths from Computational Proficiency from Wave 1 to Updating Wave 2, Patterns Wave 1 to Computational Proficiency Wave 2, and Algebra Wave 1 to Patterns Wave 2, and Algebra Wave 1 to Computational Proficiency Wave 2. The resulting model had model fit similar to the original model; however, none of the added paths attained significance. These findings suggest that the alternative paths are not tenable.

Differences in structural relationships across the two waves. To examine whether structural relationships between latent factors varied across the two waves, we re-ran Model 5 with the regression weights for the following paths constrained to equal: Patterns to Algebra (Model 5a), Computational Proficiency to Algebra (Model 5b), Updating to Patterns (Model 5c), and Updating to Computational Proficiency (Model 5d).

Inspection of the standardised estimates from models conducted for each of the two waves suggested increased dependency of Algebraic Performance on Patterns at Wave 2. The present analysis showed that models with or without the equality constraint fitted the data equally well (see Table 2 for fit indices). Because the model with the equality constraint is more parsimonious, the findings suggest that the direct relationships between Patterns and Algebra are the same across the two waves.

For Computational Proficiency to Algebra, poorer model fit was obtained when the equality constraint was applied. This suggests that there is greater reliance on Computational Proficiency in Wave 1 than in Wave 2. No significant difference in model fit was found when the equality constraint was employed for Updating to Patterns. Imposing an equality constraint on

Updating to Computational Proficiency also did not result in a poorer fit to the data. Both findings suggest that relationships with updating did not differ across the two waves, at least when other contributors to performances in Wave 2 had been taken into account.

The Role of Intelligence

Some items in patterns tasks bear semblance of those used in measures of fluid intelligence (e.g., the Raven's Progressive Colour Matrices). To examine the extent to which variance explained by the pattern tasks is due to its correlation with fluid intelligence, we conducted a further set of analysis on the Primary 4 data, to which we added a measure of fluid intelligence. Using data from the WISC block design task, we constructed a latent variable using its published reliability index to define its error variance, and added to Model 3 structural paths from Fluid Intelligence to Patterns, Computational Proficiency, and Algebra.

Inclusion of the Fluid Intelligence measure did little to improve model fit. Although the overall fit was still good, they were marginally poorer than a similar model (Model 3) that did not contain the fluid intelligence measure (see Table 2). Inspection of the structural weights revealed several findings of note. The fluid intelligence measure failed to account for a significant amount of variance in the patterns or algebraic performance measures. It did, however, predict Computational Proficiency and was also significantly correlated with Updating.

The overall pattern of relationships amongst the other latent variables remains similar to the model that did not contain the fluid intelligence measure. The only difference is that Updating no longer predicted Patterns. These findings allayed our concern that the relationship between Patterns and Algebra is an artefact of fluid intelligence. Not only did fluid intelligence failed to explain variance in algebraic performance, the relationship between patterns and algebraic performance was no poorer when fluid intelligence was added to the model.

Discussion

This study examined the relationships between updating, number patterns, computational fluency, and their respective direct and indirect relationships to algebraic proficiency. The viability of the model fitted to the Primary 4 data was tested on a second cohort of Primary 4 students and on data from the original sample who was re-administered the tests one year later. Test of equality of parameters across the two Primary 4 cohorts showed that the same structural model provided a good fit to both sets of data. The longitudinal data showed that there were differences in the extent to which variance in the algebraic measure was explained by the various predictors, but the overall patterns were similar. Both number patterns and computational proficiency had a direct relationship with algebraic proficiency. Updating showed no direct relationship with algebraic proficiency, but was directly related to number patterns and computational proficiency. As such, updating appears to have an indirect influence on algebraic proficiency via its impact on skills that underpin algebra.

Are Patterns Important?

Findings from both waves showed that patterns explained a significant amount of variance in algebraic performance even after variance common to computational skills has been controlled. Data from Primary 4 showed that removing the path between the pattern and algebraic tasks resulted in a model that produced a significantly poorer fit to the data. When data from Primary 4 and 5 were modelled separately, the findings suggested that proficiencies on the pattern and algebraic tasks strengthened from the first ($\beta = .28$) to the second wave ($\beta = .45$). However, findings from the longitudinal model revealed that the magnitude of this relationship was similar across the two waves. The difference in findings can be attributed to the difference in model specification: in the longitudinal model, Primary 4 computational proficiency was included as an additional predictor for pattern proficiency at Primary 5.

On the applied level, findings from the analyses conducted for each age group separately suggest that patterns assume more importance for children at Primary 5 than for children at Primary 4. On an explanatory level, these findings show that performance on the patterns task at Primary 5, like pattern performance at Primary 4, is closely related to computational competency at Primary 4. The longitudinal findings suggest that the apparent increase in the magnitude of dependency between patterns and algebra is attributable to this commonality. Once Primary 4 computational competency is added into the explanatory model, proficiencies in the pattern tasks per se had similar explanatory power across the two waves. The overall findings point to the importance of computational proficiency at Primary 4. It has both direct relationships with same-grade algebraic performance as well as a strong but indirect relationship on algebraic performance one year later.

Despite the importance of computational proficiency at Primary 4, the direct contribution of same-grade computation proficiency to algebraic proficiency decreased across the two waves. The decreasing role of computational proficiency is consistent with a previous study, also conducted with children in Primary 5, in which computation was found to explain no additional variance in an algebraic task when proficiencies in understanding the algebraic question and translating the question into a pictorial representation were statistically controlled (Lee et al., 2009). Together, these findings suggest that with a firm foundation at Primary 4, success in algebraic word problems becomes less directly dependent on children's fluency with higher computational skills.

However, it is important to note that this is not to say that children in Primary 5 no longer require computational proficiency in order to be successful on the algebraic task. All the algebraic problems required children to compute a solution. For this reason, ability to compute is a structural pre-requisite for success. Indeed, when we modelled a counter-intuitive hypothesis that success in algebra at Primary 4 explains variance in Primary 5 computational fluency, the path was non-significant. These findings suggest that the degree to which children are reliant on computational skills specific to Primary 5 is less than the extent to which they relied on their computational skills one year ago.

One somewhat surprising finding was that though the patterns tasks at Primary 4 and 5 were closely correlated on a bivariate level, data from the longitudinal model showed no direct relationship. The pattern of findings suggests that the relationship between the pattern tasks is largely explained by their mutual correlation with computational proficiency at Primary 4. This finding is consistent with Orton and Orton's (1999) findings and suggests that at both grade levels, performances in the pattern tasks are critically dependent on early computational competence.

What are the pattern skills that are important for solving algebraic questions? Although our data show a close relationship between the two, they do not allow us to make definitive statements on the identity of these skills. One question for future study is to consider whether and which of these processes, recognizing part-whole, doing-undoing of operations, detecting sameness and differences, and identifying the structure underpinning each task (Mason, 1996), provide a critical link between patterns and algebra.

The Role of Updating

Previous studies identified a close relationship between updating and algebraic skills (Lee et al., 2004; Lee et al., 2009), but had not considered whether this relationship was mediated by the relationships between updating and the component skills needed to solve algebraic problems. The present findings show that updating is strongly predictive of both patterns and computational proficiencies. Though updating capacity was correlated with performance on the algebraic problems, the contribution of updating was fully mediated by patterns and computational fluency.

Why did updating fail to have a direct effect on algebraic performance? One possibility is that computation and pattern explained most of the variance in algebraic performance. However, this does not seem to be the case; data from both Primary 4 and 5 showed that only 63 to 68% of variance in algebraic performance was explained. An alternative explanation is suggested by some previous findings. In Lee et al. (2009), accuracy in the algebraic task was predicted by children's ability to translate text based specification of quantitative relations, e.g., more than, less than, into appropriate mathematical operators: addition and subtraction. Variation in proficiency on this translational process was predicted by working memory capacity. Furthermore, they found working memory failed to account for additional variance in their algebraic problem task (on which the present task is based), after the contributions of other componential tasks were controlled. Considered together, it is likely that updating provides the resources for the various components that are executed when children solve an algebraic problem. The lack of a direct effect suggests that the coordination or planning of such tasks does not impose additional resource requirements.

Given the complexity of algebraic problems, this conclusion seems counter-intuitive. However, it has to be borne in mind that such problems, though complex, are often drawn from a relatively small set of standardised forms (Mayer, 1981). In some classrooms, children are taught to approach these problems in a procedural manner. Though the constituent tasks that allow children to be successful may require much effort, the type and sequence of tasks that need to be performed -- e.g., first figure out the generator, then the mathematical operations involved, draw a model, perform undoing operation -- may have been automatized to such an extent that they do not require extensive thoughts or planning.

Conclusions and Caveat

Are patterns important for algebraic problem solving? As far as we are aware, this is the first study to provide direct empirical evidence on this question. Our findings identified proficiency in number patterns as being significantly correlated with algebraic performance. Furthermore, exclusion of pattern proficiency from our model produced a significantly poorer fit to the data. Even after controlling for variation in computational skills, being able to recognise patterns, in and of itself, is correlated with algebraic proficiency. Solving a patterns task like those presented here requires the child to use computation to realise the rule and complete the missing values. It may be that having to understand the relationship stated in an algebraic problem shares commonality with having to ascertain the relationship in a patterns task. Nonetheless, it should be noted that proficiency in the pattern tasks, though important, was not the best predictor of algebraic performance. Both updating and computational proficiency contributed to proficiency on the patterns tasks and thus had an indirect effect on algebraic performance. An examination of total effects showed both updating and computational proficiency had greater total effect on algebraic performance than did pattern.

As with most complex structural equation models, there exists equivalent models that provide identical statistical fit to the observed data as our favoured model. We used the longitudinal data to test the viability of the most promising relationships specified by the equivalent models. Because none of these paths attained significance, they gave us additional confidence on the viability of the proposed model.

Findings regarding the relationship of curricular components are likely to be sensitive to the content of specific curriculum. As noted in the report from the US National Advisory Panel on Mathematics (2008), not all countries include pattern tasks in their K-6 curriculum. Indeed, algebraic tasks are not introduced in some countries until the secondary school years. What our longitudinal data show is that within the context of our local curriculum, prior exposure to patterns at Primary 4 has only an indirect effect on proficiency on algebraic performance at Primary 5. For the older children, the within grade relationship between pattern and algebraic performance is stronger. A teaching experiment that manipulates exposure to patterns will provide a more direct examination of its effect on algebraic proficiency.

Pedagogical implications

In Singapore, primary children are taught to use the model method to solve algebraic type word problems (Ng & Lee, 2009). This method provides an avenue to describe the problem before the process of calculating for the solution begins. This process of “first-describing-and-then-calculating” (Post, Behr, & Lesh, 1988) may make recognition of patterns particularly important in the first stage of describing or representing the relationships. Once the relationship has been identified and represented, the child needs to capitalize on their facility with arithmetic operations and number facts to complete the problem. For children using the model method, the pictorial representation serves as an external visual referent displaying the relations in the problem. This may have the effect of lowering the updating demands of the algebraic tasks. There are, of course, other methods to teach and represent algebraic problems other than the model method, and it may be that different methods draw upon updating, pattern, and computational skills to different extents. However, it seems clear that algebraic reasoning will be difficult if the child has either poor facility with computation or has poor ability to recognize patterns in information and generalize rules about those patterns.

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The study reported in this paper was supported by a grant from the Office of Educational Research, CRP8/05KL. We thank the children who participated in the study and school administrators who provided access and assistance. Views expressed in this article do not necessarily reflect those of the National Institute of Education.

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Table 1.

Descriptive Statistics and Correlations for P4 (in the upper row/diagonal) and P5

	Variable	*M*	*SD*	1	2	3	4	5	6	7	8	9	10
1	Mr. X	15.59 / 17.91	5.39 / 5.84		.30**	.31**	.26**	.12	.18*	.29**	.17*	.16	.17*
2	Listening Recall Memory Score	11.75 / 13.07	3.42 / 3.36	.27**		.41**	.29**	.24**	.23**	.27**	.32**	.27**	.30**
3	Pictorial Updating	81.68 / 87.29	10.76 / 10.39	.26**	.35**		.37**	.26**	.30**	.43**	.41**	.40**	.46**
4	WIAT Number Ops Group 1	9.92 / 10.41	1.40 / 1.58	.29**	.24**	.44**		.46**	.50**	.55**	.55**	.50**	.55**
5	WIAT Number Ops Group 2	9.83 / 10.36	1.11 / 1.54	.22**	.22*	.39**	.67**		.59**	.42**	.45**	.38**	.43**
6	WIAT Number Ops Group 3	9.76 / 10.78	1.31 / 1.55	.27**	.21*	.39**	.65**	.67**		.45**	.45**	.44**	.44**
7	Number Patterns	8.13 /	3.68 /	.42**	.21*	.52**	.60**	.42**	.45**		.64**	.60**	.69**

		4.59	2.38										
8	Function Machine (Output)	19.87 / 13.49	8.50 / 7.13	.35**	.32**	.47**	.54**	.49**	.52**	.67**		.91**	.95**
9	Function Machine (Rule)	19.18 / 13.01	9.38 / 6.69	.30**	.26**	.45**	.59**	.54**	.50**	.68**	.88**		.91**
10	Function Machine (Input)	18.68 / 14.02	9.49 / 7.05	.34**	.27**	.47**	.59**	.49**	.53**	.71**	.97**	.90**	
11	Algebra Group 1	1.63 / 1.39	1.62 / 1.27	.27**	.15	.38**	.57**	.51**	.55**	.63**	.65**	.60**	.67**
12	Algebra Group 2	1.25 / 1.05	1.21 / 1.05	.24**	.23*	.41**	.56**	.48**	.48**	.64**	.63**	.62**	.65**
13	Algebra Group 3	1.25 / 1.06	1.29 / 1.00	.25**	.19*	.37**	.43**	.44**	.46**	.50**	.56**	.52**	.53**

	Variable	11	12	13
1	Mr. X	.21*	.23**	.21*
2	Listening Recall Memory Score	.22**	.19*	.22**
3	Pictorial Updating	.36**	.33**	.34**
4	WIAT Number Ops Group 1	.58**	.54**	.54**
5	WIAT Number Ops Group 2	.49**	.43**	.48**
6	WIAT Number Ops Group 3	.52**	.52**	.49**
7	Number Patterns	.68**	.63**	.65**
8	Function Machine (Output)	.63**	.60**	.61**
9	Function Machine (Rule)	.56**	.54**	.51**
10	Function Machine (Input)	.64**	.60**	.61**
11	Algebra Group 1		.89**	.89**
12	Algebra Group 2	.78**		.87**
13	Algebra Group 3	.65**	.59**	

Note: **p < .01, ***p < .001.

Table 2

Fit Indices for Structural Models

	χ^2	*Df*/FP	CFI	RMSEA/ PCLOSE	AIC	$\Delta\chi^2$	Δdf	*p*
Primary 4								
Model 1	90.59**	59/32	.98	.06/.25	180.59			
Model 2	97.08**	60/31	.98	.06/.16	185.08			
Model 3	90.67**	60/31	.98	.06/.27	178.67			
Model 4	105.16***	61/30	.97	.07/.08	191.16			
Model 1 vs. Model 2						6.49	1	.01
Model 1 vs. Model 3						.08	1	.77
Model 3 vs. Model 4						14.49	1	<.01
Model 3 with fluid intelligence measure	117.58***	70/35	.97	.07/.09	215.58			
Testing for Factorial Invariance with a Primary 4 Replication Sample using Model 3								
Unconstrained	222.16***	120/62	.97	.05/.30	416.35			
Factor loading	238.17***	129/53	.96	.05/.30	412.49	16.01	9	.07
Structural weights[a]	240.00***	134/48	.96	.05/.40	403.30	1.83	5	.87
Structural covariance[a]	241.96***	135/47	.96	.05/.40	403.04	1.96	1	.16
Disturbances[a]	245.70***	138/44	.96	.05/.42	400.16	3.74	3	.29
Measurement errors[a]	286.08***	151/31	.95	.06/.21	411.86	40.38	13	<.01
Primary 5								
Model 1	99.63**	59/32	.97	.07/.12	187.65			
Model 2	108.48***	60/31	.96	.07/.05	196.48			
Model 3	99.65**	60/31	.97	.07/.12	187.65			

Model 4	103.65**	61/30	.97	.07/.09	189.65			
Model 1 vs. Model 2						8.85	1	<.01
Model 1 vs. Model 3						.02	1	.89
Model 3 vs. Model 4						4.00	1	<.05
Primary 4 to 5								
Location of equality constraints across the two waves								
Model 5 (unconstrained)	391.40***	273/78	.96	.05/.30	599.40			
Model 5a (Patterns to Algebra)	391.48***	274/77	.96	.05/.31	597.48			
Model 5b (Computational Proficiency to Algebra)	409.01***	274/77	.96	.06/.15	615.01			
Model 5c (Updating to Patterns)	391.90***	274/77	.96	.05/.31	597.90			
Model 5d (Updating to Computational Proficiency)	394.40***	274/77	.96	.05/.28	600.40			
Model 5 vs. Model 5a						.08	1	.77
Model 5 vs. Model 5b						17.61	1	<.01
Model 5 vs. Model 5c						.50	1	.48
Model 5 vs. Model 5d						3.00	1	.08

Note: **p < .01, ***p < .001. [a] Models used for testing equality constraints were cumulative: each step included constraints applied to earlier models. FP = number of free parameters to be estimated (excluding intercepts).

Table 3

Unstandardized Parameter Estimates of Structural Relationships between Updating, Computational, Patterns and Algebraic Proficiencies across Different Models

	Models			
Relationships	1	2	3	4
	Primary 4			
Updating to Patterns	.27*	.16	.28*	1.03***
Updating to Computational Proficiency	.08***	.08***	.08***	.12***
Updating to Algebra	< -.01	< -.01	---	---
Computational Proficiency to Patterns	4.89***	6.08***	4.87***	---
Computational Proficiency to Algebra	.69***	1.00***	.67***	.62***
Patterns to Algebra	.04**	---	.04**	.04***
	Primary 5			
Updating to Patterns	.09*	.41*	.09*	.21***
Updating to Computational Proficiency	.12***	.19***	.12***	.16***
Updating to Algebra	< -.01	.17*	---	---
Computational Proficiency to Patterns	.57**	-.78	.58**	---
Computational Proficiency to Algebra	.43***	< -.01	.42***	.42***
Patterns to Algebra	.31***	---	.31***	.31***

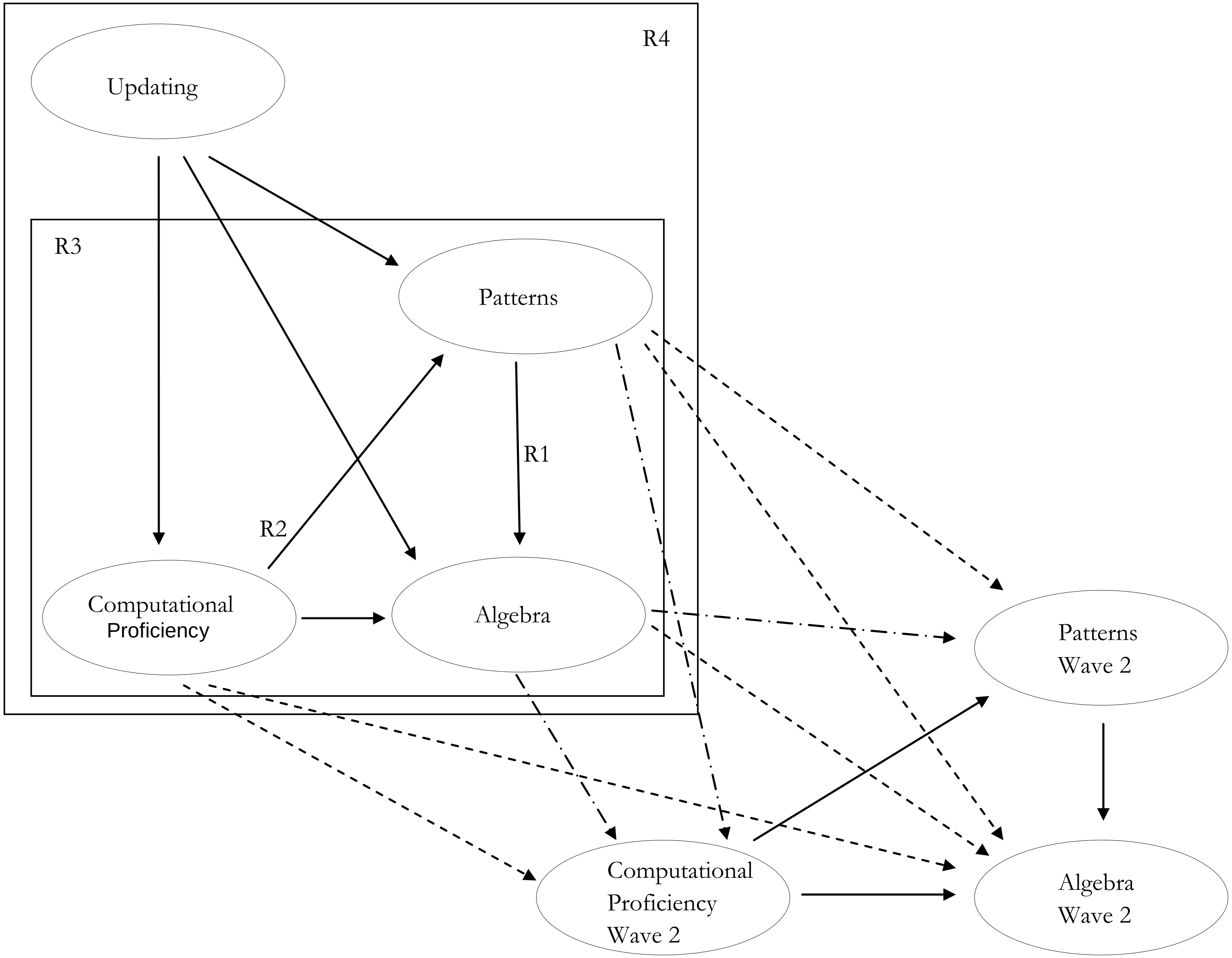
Note. * ≤ .05, **p ≤ .01, ***p ≤ .001.

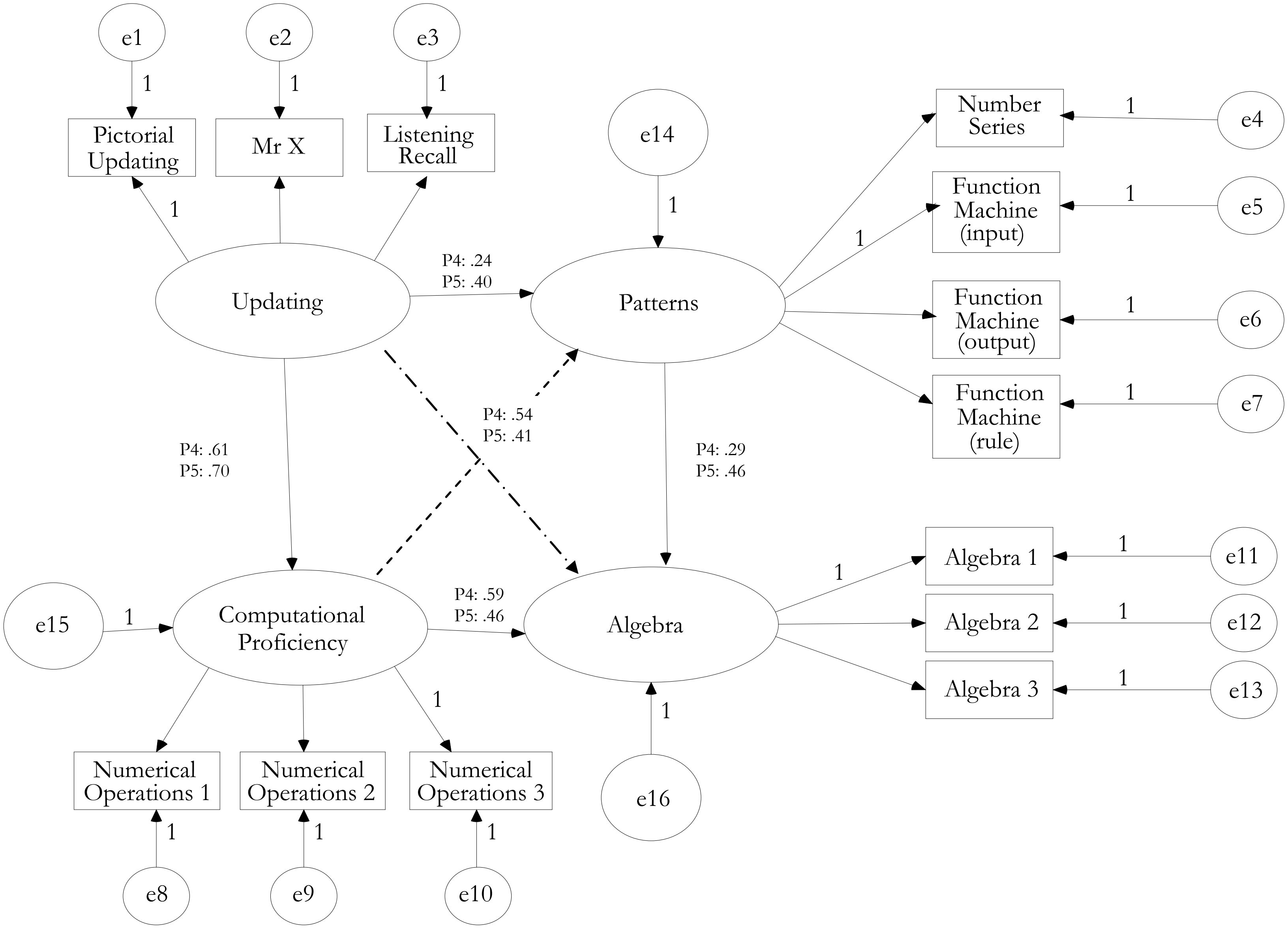
Figure Captions

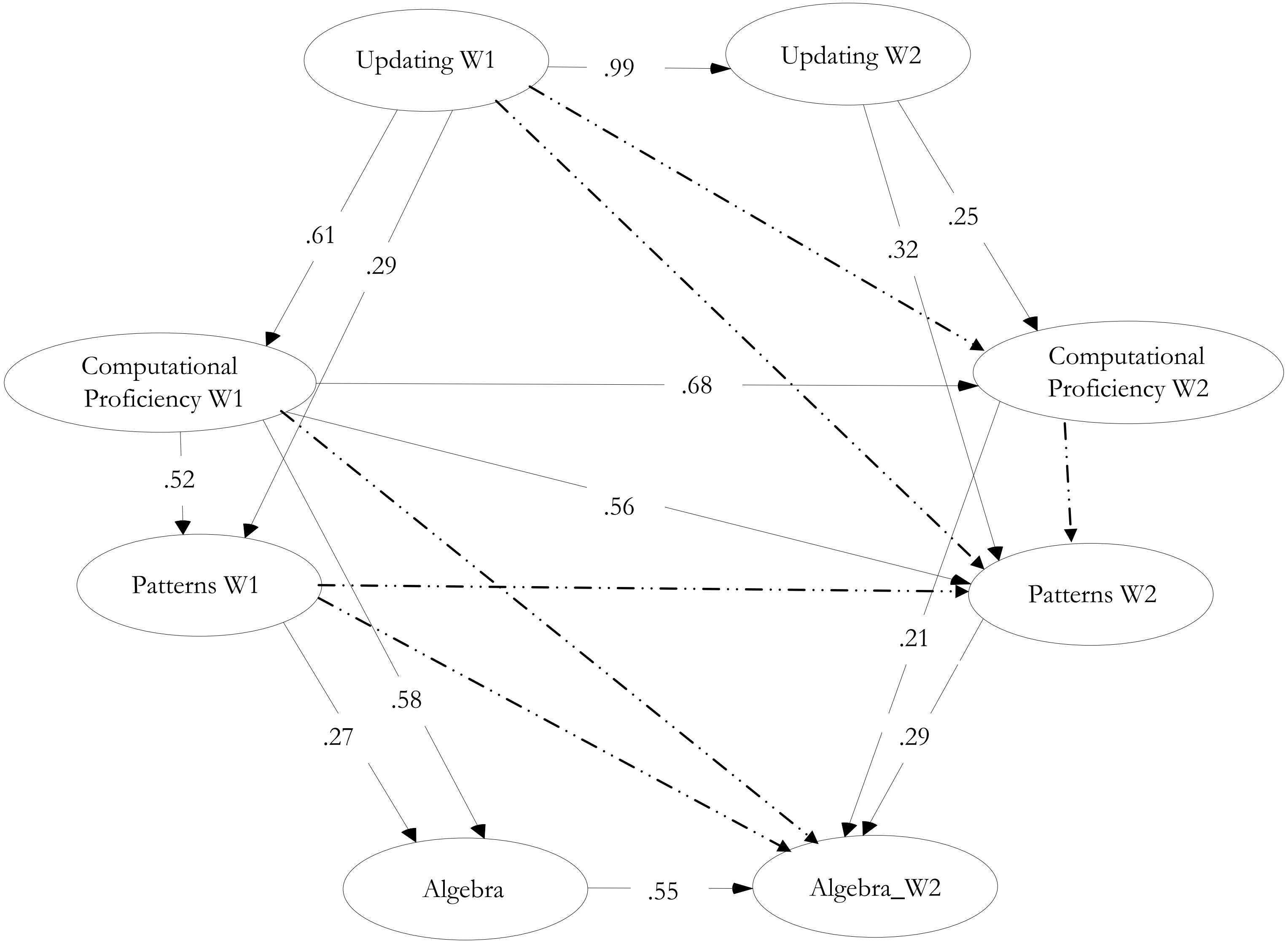
Figure 1. Relationships under evaluation. Relationships within the larger rectangle are evaluated using data from Primary 4. Relationships represented by the dashed lines are evaluated using the longitudinal data. For R1, we examined the relationship between patterns and algebra. R2 examined the relationship between computational proficiency and patterns. R3 examined simultaneously the relationships between computational proficiency, patterns, and algebra. R4 focused on the relationships between the updating and the mathematical measures. Dash-and-dot lines refer to alternative and equivalent paths.

Figure 2. Structural equation model of the relationships between updating, computational, and patterns proficiencies on algebraic performance. In Model 1, all paths were included. In Model 2, direct path from patterns to algebra (bold line) was constrained to zero. In Model 3, direct path from updating to algebra (dash-and-dot line) was constrained to zero. In Model 4, direct paths from updating to algebra and computational proficiencies to patterns (dashed line) were constrained to zero. Models for Primary 4 and 5 were estimated separately. Values are standardized estimates of the final model (Model 3).

Figure 3. Structural longitudinal model of the relationships between updating, computational, and patterns proficiencies on algebraic performance. Dash-and-dot lines represent relationships that were not significant. Values are standardized path coefficients of the final model (Model 5).


R4
Updating
R3
Patterns
R1
R2
Computational Proficiency
Algebra
Patterns Wave 2
Computational Proficiency Wave 2
Algebra Wave 2


e1
e2
e3
1
1
1
Pictorial Updating
Mr X
Listening Recall
e14
1
Number Series
e4
1
Function Machine (input)
e5
1
1
Updating
P4: .24
P5: .40
Patterns
Function Machine (output)
e6
1
Function Machine (rule)
e7
1
P4: .54
P5: .41
P4: .61
P5: .70
P4: .29
P5: .46
Algebra 1
e11
1
1
Algebra 2
e12
1
Algebra 3
e13
1
e15
1
Computational Proficiency
P4: .59
P5: .46
Algebra
1
1
Numerical Operations 1
Numerical Operations 2
Numerical Operations 3
1
1
1
e16
e8
e9
e10


Updating W1
.99
Updating W2
.61
.29
.32
.25
Computational Proficiency W1
.68
Computational Proficiency W2
.52
.56
Patterns W1
Patterns W2
.21
.58
.27
.29
Algebra
.55
Algebra_W2